Recall: integrating functions over curves

Let C be a smooth curve in $\mathbb{R}^3$ parametrized by $\mathbf{r}(t)$, $a \leq t \leq b$. Then

$$\text{Length}(C) = \int_a^b |\mathbf{r}'(t)| dt.$$

Furthermore, if g is a continuous function on C, then

$$\int_C g \; ds = \int_a^b g(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

Today: Integrate a function over a surface (or estimate the integral algebraically or geometrically).

Applications: Find surface area, mass, average value.

Practice with surface area

Find the surface area of $S = \{x^2 + y^2 + z^2 = 1\}$.

Step 1: Parametrize S

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle,$$
$$0 \le \phi \le \pi; \qquad 0 \le \theta \le 2\pi.$$

Step 2: Calculate $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$.

$$\mathbf{r}_\phi(\phi, \theta) = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle;$$
$$\mathbf{r}_\theta(\phi, \theta) = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle.$$

So

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix} \\
&= \mathbf{i}(\sin^2\phi\cos\theta) - \mathbf{j}(-\sin^2\phi\sin\theta) \\
&\qquad + \mathbf{k}(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) \\
&= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{\sin^4\phi(\cos^2\theta + \sin^2\theta) + \sin^2\phi\cos^2\phi} \\
&= \sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} \\
&= \sqrt{\sin^2\phi} = \sin\phi
\end{aligned}$$

(since $\sin\phi \geq 0$ on D).

So the surface area of the sphere is

$$\iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA = \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta$$
$$= 2\pi \left[-\cos\phi\right]_0^{\pi}$$
$$= 4\pi.$$

Practice with surface area

Consider a can with sides given by the cylinder $\{x^2 + y^2 = 1, -1 \leq z \leq 1\}$, parametrized by

$$\mathbf{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle,$$
$$0 \leq \theta \leq 2\pi, \qquad -1 \leq z \leq 1.$$

Find the surface area of the can. (Don't forget the top and bottom!)

(a) 2π

(b) 4π

(c) 6π

(d) 8π

(e) I don't know how.

Solution

First let's calculate the area of the cylinder, and then we'll add the top and bottom.We use the parametrization given above,

$$\mathbf{r}(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle,$$
$$0 \le \theta \le 2\pi, \qquad -1 \le z \le 1.$$

Step 2: Find $|\mathbf{r}_\theta \times \mathbf{r}_z|$.

$$\mathbf{r}_\theta(\theta, z) = \langle -\sin\theta, \cos\theta, 0\rangle;$$
$$\mathbf{r}_z(\theta, z) = \langle 0, 0, 1\rangle.$$

So

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta,$$

and $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$.

So the surface area of the cylinder is

$$\int_0^{2\pi} \int_{-1}^{1} dz d\theta = 4\pi.$$

The top and the bottom of the can are each circles of radius one and hence have (surface) area π, so the total area is

$$4\pi + \pi + \pi = 6\pi.$$