Recall: Integrating functions on curves.

- Let C be a curve in $\mathbb{R}^3$, parametrized by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$.

  Then $\text{length}(C) = \int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt$.

- More generally, for f a continuous function on C,

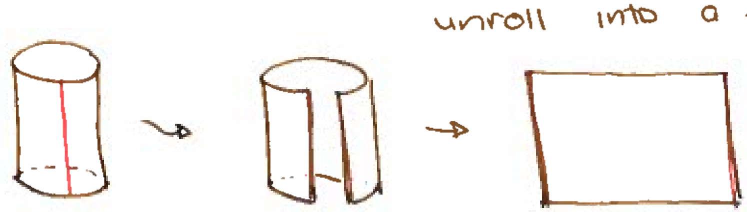
$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt.$$

Today: integrate functions on surfaces.

Example: Find the surface area of S, where S is

(a) a cube - easy: cut into 6 faces; measure each face.

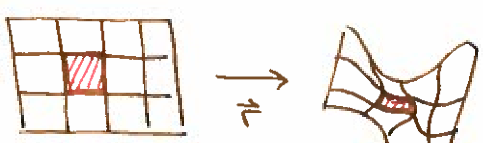
(b) a cylinder - pretty easy: slice down the side; and unroll into a flat rectangle.



(c) a sphere - hard

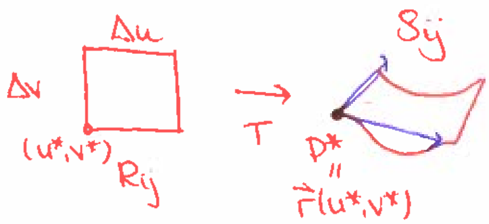
General strategy:
- cut the surface into small pieces ("patches")
- approximate each patch by a small parallelogram.
- add up the area of the parallelograms.

More precisely: Suppose S is parametrized by $\vec{r}(u,v)$, $(u,v) \in D \subset \mathbb{R}^2$.



D S

- A small rectangle R_{ij} in D gets mapped to a patch S_{ij} in S



The patch S_{ij} is approximated by the parallelogram with sides $\Delta v\, \vec{r}_v(u^*,v^*)$ and $\Delta u\, \vec{r}_u(u^*,v^*)$

$$\Rightarrow \text{Area}(S_{ij}) \approx |\Delta u\, \vec{r}_u(u^*,v^*) \times \Delta v\, \vec{r}_v(u^*,v^*)|$$

$$= |\vec{r}_u(u^*,v^*) \times \vec{r}_v(u^*,v^*)|\, \Delta u \Delta v.$$

$$\therefore \text{Area}(S) \approx \sum_{i=1}^{n}\sum_{j=1}^{m} |\vec{r}_u(u_i^*,v_j^*) \times \vec{r}_v(u_i^*,v_j^*)|\, \Delta u \Delta v.$$

↑ Riemann sum.

Theorem: Let S be a smooth surface paramerized by

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, $(u,v) \in D \subset \mathbb{R}^2$.

(Assume S is (mostly) covered exactly once by $\vec{r}$)

Then surface area of $S = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$.

Example: Find the surface area of $S = \{x^2+y^2+z^2=1\}$

Step 1 - paramerize S.

$\vec{r}(\phi,\theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

Step 2 - calculate $|\vec{r}_\phi \times \vec{r}_\theta|$.

- $\vec{r}_\phi(\phi,\theta) = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$
- $\vec{r}_\theta(\phi,\theta) = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$$\Rightarrow \vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= \vec{i}(\sin^2\phi\cos\theta) - \vec{j}(-\sin^2\phi\sin\theta) + \vec{k}(\cos\phi\sin\phi\cos^2\theta + \cos\phi\sin\phi\sin^2\theta)$$

$$= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi\rangle$$

$$\Rightarrow |\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \cos^2\phi\sin^2\phi}$$

$$= \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi}$$

$$= \sin\phi \quad (\text{since } \sin\phi \geq 0 \text{ for } 0 \le \phi \le \pi.$$

Step 3: Surface area $= \iint_D |\vec{r}_\phi \times \vec{r}_\theta| \, dA$.

$\Rightarrow \iint_D \sin\phi \, dA = \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta$

$= 2\pi \left[-\cos\phi\right]_0^{\pi} = 4\pi$. (Check: this makes sense)

Example 1: Consider the can with sides given by the cylinder $\{x^2+y^2=1, \ -1 \le z \le 1\}$.

parametrized by $\vec{r}(\theta, z) = \{ \langle \cos\theta, \sin\theta, z \rangle$
$0 \le \theta \le 2\pi, \ -1 \le z \le 1$.

Find the surface area of the can (don't forget top & bottom!)
[See solution on slides.]

- Let S be a surface in $\mathbb{R}^3$ parametrized by $\vec{r}(u,v)$, $(u,v) \in D$ as before.

  Let f be a continuous function on S.

$\Rightarrow$ the integral of f over S is $\iint_S f \, dS = \iint_D f(\vec{r}(u,v)) \cdot |\vec{r}_u \times \vec{r}_v| \, dA$.

Geometric interpretations:

- Given a piece of sheet metal with shape S and density at $(x,y,z) \in S$ given by $\rho(x,y,z)$ (in unit mass/unit area),

  total mass $= \iint_S \rho \, dS$.

  center of mass, moment of inertia, etc....

- $\iint_S f \, dS = (\text{surface area of } S) \cdot (\text{average value of } f \text{ on } S)$.

Example: Find average value of $f(x,y,z) = xy + z$ over the surface S which is the piece of the cone $x^2+y^2=z^2$ with $0 \le z \le 1$.

Step 1: parametrise S:

$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$, $\quad 0 \le u \le 2\pi$, $\quad 0 \le v \le 1$

Step 2: Find $|\vec{r}_u \times \vec{r}_v|$.

$\vec{r}_u(u,v) = \langle -v\sin u, v\cos u, 0\rangle$

$\vec{r}_v(u,v) = \langle \cos u, \sin u, 1\rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = \vec{i}\, v\cos u - \vec{j}(-v\sin u) + \vec{k}(-v)$$

$$= \langle v\cos u, v\sin u, -v\rangle$$

$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{v^2+v^2} = \sqrt{2}\, v$, since $v \geq 0$.

Step 3: ~~A~~ Find surface area $= \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| dA$

$$\int_0^{2\pi}\int_0^1 \sqrt{2}\, v \, dv \, du = 2\pi\left[\frac{\sqrt{2}}{2}v^2\right]_0^1 = \sqrt{2}\pi .$$

Step 4: Find integral $\iint_S f dS$: $\quad f(x,y,z) = xy + z$

$$\int_0^{2\pi}\int_0^1 (v^2\cos u\sin u + v)\sqrt{2}v \, dv\, du$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 v^3\cos u\sin u + v^2 \, dv\, du$$

$$\cancel{= \sqrt{2}\int_0^{2\pi}[\;]} \quad = \sqrt{2}\int_0^1\int_0^{2\pi} v^3\cos u\sin u + v^2 \, du\, dv$$

$$= \sqrt{2}\int_0^1\left[\frac{v^3}{2}\sin^2 u + v^2 u\right]_0^{2\pi} dv$$

$$= \sqrt{2}\int_0^1 2\pi v^2 \, dv = \sqrt{2}\left[\frac{2\pi}{3}v^3\right]_0^1 = \frac{2\sqrt{2}\pi}{3}.$$

Step 5: Average $=$ Integral / surface area

$$\text{Average} = \frac{2\sqrt{2}\pi}{3} \Big/ \sqrt{2}\pi = 2/3 .$$