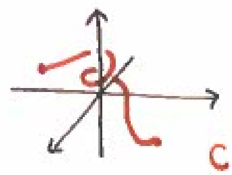
1 Parametrizations of a line.

- Let x, y, z be continuous real-valued functions on $[a,b]$.
  - ↳ the set $C = \{(x(t), y(t), z(t)) \mid t \in [a,b]\}$ is called a **curve**
    - it's a set of points in $\mathbb{R}^3$.

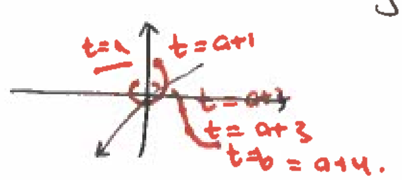


This is like tire tracks in snow

↳ the vector-valued function $\vec{r} : [a,b] \to V_3$ (vectors in $\mathbb{R}^3$)
$t \mapsto \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

is a **parametrization of C**

- it gives instructions for moving along C

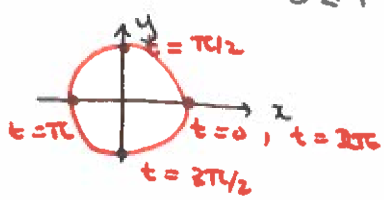


This is like a movie.

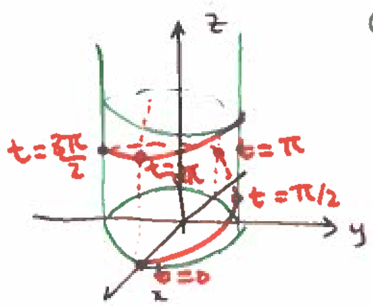
- One curve has many parametrizations.
  - you could go twice as quickly, go in the opposite direction, start at a different time, speed up or slow down.
- Similar definitions in $\mathbb{R}^2$. (plane curves vs. space curves)

Example - Graph the curve parametrized by $\vec{r}(t) = \langle \cos t, \sin t \rangle$
$0 \le t \le 2\pi$.

↑ Sorry, my pen malfunctioned!



Example - Graph the curve parametrized by $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$
$0 \le t \le 8\pi$



→ Helix!

[see slides for a better picture]

2

Terminology

Let $\vec{r}(t)$ parametrize a curve C.

Let $\vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, $P = (x_0, y_0, z_0) \in C$.

the **derivative of $\vec{r}$** is

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \lim_{h\to 0} \frac{\langle x(t+h) - x(t), y(t+h) - y(t), z(t+h) - z(t) \rangle}{h}$$

$$= \langle x'(t), y'(t), z'(t) \rangle.$$

the **tangent vector** to C at P is $\vec{r}\,'(t_0)$.

(Length depends on choice of parametrization $\vec{r}$, but direction doesn't (up to sign)).

↳ the **unit tangent vector** is $\vec{T}(t_0) = \dfrac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|}$

the **tangent line to C at P** is the line L through $P = (x_0, y_0, z_0)$ and parallel to $\vec{r}\,'(t_0)$:

$$L(t) = \langle x_0, y_0, z_0 \rangle + (t-t_0)\,\vec{r}\,'(t_0)$$

$$= \langle x_0 + x'(t_0)(t-t_0),\ y_0 + y'(t_0)(t-t_0),\ z_0 + z'(t_0)(t-t_0) \rangle.$$

(equation depends on choice of $\vec{r}$, but the line itself doesn't)

the **linear approximation to $\vec{r}$ near t_0** is

$$\vec{r}(t_0 + \Delta t) \approx \vec{r}(t_0) + (\Delta t)\,\vec{r}\,'(t_0).$$

Physical interpretation:

- If $\vec{r}(t)$ denotes the position of a particle at time t:
  - $\vec{r}\,'(t) = \vec{v}(t)$ is the velocity

    $|\vec{v}(t)| = |\vec{r}\,'(t)|$ is the speed
  - $\vec{r}\,''(t) = \vec{a}(t)$ is the acceleration.

Example. Find the velocity, speed, and acceleration if

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

again!

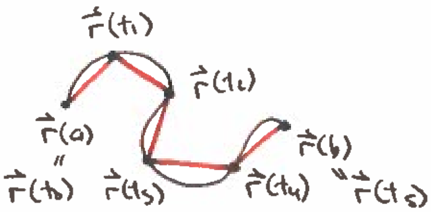
- $\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

  $\Rightarrow$ speed $= |\vec{v}(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = \sqrt{1} = 1$.
- $\vec{a}(t) = \vec{r}\,''(t) = \langle -\cos t, -\sin t, 0 \rangle$ ← points to the centre of the cylinder.

§ ARC LENGTH (§13.3)

- Distance = Speed · time.
- Suppose $\vec{r}(t)$ is the position of a particle at time t, $t \in [a,b]$.
  - at time t, how far has the particle travelled?
  - what is the total distance travelled from a to b?

    i.e. what is the length of the curve?



- divide $[a,b]$ into ~~five~~ n intervals of equal size $\frac{b-a}{n} = \Delta t$

$$L \approx \sum_{i=1}^{n} \frac{|\vec{r}(t_i) - \vec{r}(t_{i-1})|}{\Delta t} \Delta t$$

As $n \to \infty$, • $\Delta t = \frac{b-a}{n} \to 0$.

• $\left|\frac{\vec{r}(t_i) - \vec{r}(t_{i-1})}{\Delta t}\right| \to |\vec{r}\,'(t_{i-1})|$

So $L = \int_a^b \underbrace{|\vec{r}\,'(t_i)|}_{\text{speed}} \cdot \underbrace{dt}_{\text{time}}$

Theorem: Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $t \in [a,b]$.

The length L of the curve parametrized by $\vec{r}$ is

L = distance travelled $= \int_a^b (\text{speed at time } t)\, dt$

$$= \int_a^b |\vec{r}\,'(t)|\, dt$$

$$= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.$$

Example: Find the length of the helix $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 8\pi$.

$$L = \int_0^{8\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \int_0^{8\pi} \sqrt{2}\, dt$$

$= 8\sqrt{2}\,\pi$.

WARNING: Don't trace over the path more than once!

16.4

$\boxed{2}$ Find the length of $\vec{r}(t) = \langle t, \sqrt{1-t^2} \rangle$, $-1 \le t \le 1$

Three methods :

- from the definition
- from the picture
- reparametrize