From Wednesday: (limits)

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$\lim_{x \to 0} f(x,0) = 1/2, \qquad \lim_{y \to 0} f(0,y) = 1/2.$$

[L] What can we say about $\lim_{(x,y)\to(0,0)} f(x,y)$?

Definitions: $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$*$ f is continuous on D if it is continuous at (a,b) for all $(a,b) \in D$.

↳ So if f is continuous, we can find its limits by evaluating at the desired points.

Examples of continuous functions:

- polynomials, sin, cos
- products, sums of continuous functions
  - quotients of continuous functions, wherever the denominator isn't zero.
- composition of continuous functions

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f"] \arrow[rr, bend right, "h"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$\Rightarrow$ $h = g \circ f : (x,y) \mapsto g(f(x,y))$ is continuous.

Exceptions:

- $f(x) = \frac{1}{x}$ continuous for $x \neq 0$
- $f(x) = x^{1/n}$ continuous for $x \geq 0$.
- $f(x) = \ln(x)$ continuous for $x > 0$.

Example: Where is $f(x,y,z)$ continuous?

$$f(x,y,z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2}$$

◁- continuous for $y \geq 0$

◁- always continuous.

But we can't have $x^2 - y^2 + z^2 \neq 0$.

$\Rightarrow$ f is continuous on $D = \{(x,y,z) \mid y \geq 0,\ x^2 - y^2 + z^2 \neq 0\}$

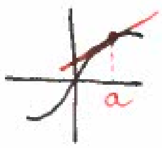
[L] Find $\lim_{(x,y,z)\to(0,1,0)} f(x,y,z)$.

(§ 14.3) PARTIAL DERIVATIVES

Recall from Calc I: for $f: \mathbb{R} \to \mathbb{R}$, we have

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(a)$ gives a line that approximates the graph near $(a, f(a))$



↳ tangent line at a is

$$y = f(a) + \underbrace{f'(a)}_{\text{slope}}(x - a)$$

Warning: $f'(a)$ may not exist.

e.g. • $f(x) = |x|$ has no derivative at $x = 0$.

• Brownian motion has no derivative anywhere!

Partial derivative: Fix $f: \mathbb{R}^2 \to \mathbb{R}$.

$\frac{\partial f}{\partial x}$ measures how f changes if we fix the y value and vary only the x-value.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}.$$

Not hard to calculate: treat y like a constant and use Calc 1.

Example: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 2xy^3$$

e.g. $\frac{\partial f}{\partial x}(1,0) = 3(1)^2 + 2(1)(0)^3 = 3.$

Notation $\frac{\partial f}{\partial x}, \frac{\partial}{\partial x} f, f_x, f_1, D_1 f.$

↖ first variable.

Similarly. $$\frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}.$$

$\frac{\partial f}{\partial a}(a,b,c,d) = \lim$ ↳ works for any number of variables.

Example: Calculate f_x if $f(x,y) = \sin(3x + xy)$.

$\boxed{i}$ **Rmk.** $f_y = \cos(3x + xy)\cdot x$

§ HIGHER DERIVATIVES.

Since $f_x = \frac{\partial f}{\partial x}$ is again a function $\mathbb{R}^2 \to \mathbb{R}$,

we can take its partial derivatives again, with respect to x or y:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y x}\right) = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

Second order partial derivatives

differentiate in x first then in y.

and again... f_{xyx}, f_{xyx} etc.
(again, may not always be defined).

Example: Calculate f_{xy} $\boxed{i}$

Rmk: $f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x}(\cos(3x + xy)\cdot x)$

$$= \left[\frac{\partial}{\partial x}(\cos(3x+xy))\, x\right] + \cos(3x + xy)$$

$$= -\sin(3x + xy)(3+y)x + \cos(3x + xy).$$

What do you get? $\boxed{i}$

Clairaut's Theorem: If $f: \mathbb{R}^2 \to \mathbb{R}$ is such that f_{xy} and f_{yx} are continuous on $D \subset \mathbb{R}^2$, then $f_{xy} = f_{yx}$ on D.

WARNING: It can fail if ~~~~ f_{xy} or f_{yx} aren't continuous.

Example: $f(x,y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

Fact: $f_{xy}(0,0) = 1$; $f_{yx}(0,0) = -1$.

Continuity example:

$f:\mathbb{R}^2 \to \mathbb{R}$ given by: $f(x,y) = \begin{cases} x^2/\sqrt{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

- f is continuous for $(x,y) \neq (0,0)$ because

  x^2 is continuous

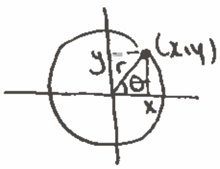
  $\sqrt{x^2+y^2}$ is continuous and non-zero.

But what about at $(x,y) = (0,0)$?

Recall: f is continuous at $(0,0)$

$\iff \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$ ↖ 0.

Polar coordinates:



$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$$f(r\cos\theta,\ r\sin\theta) = \frac{r^2\cos^2\theta}{r} = r\cos\theta.$$

$$-1 \leq \cos\theta \leq 1$$

$$\Rightarrow -r \leq f(r\cos\theta,\ r\sin\theta) \leq r.$$

$\Rightarrow$ By the Squeeze Theorem, $\lim\limits_{r\to 0} f(r\cos\theta,\ r\sin\theta) = 0.$

So f is continuous everywhere!