MATH 402 Worksheet 6

Wednesday 14 November, 2018

Exercise 1: Pasch's Axiom for Omega-Triangles, Part 2
In this exercise, we will prove the following:

Theorem 1. *Let $PQ\Omega$ be an omega-triangle. Let ℓ be a line which passes through one of the sides, but not through a vertex P, Q or Ω. Then ℓ must pass through exactly one of the other two sides.*

We will use Pasch's Axiom for ordinary triangles, as well as Part 1 of Pasch's Axiom for Omega-Triangles to prove the result. If you need to, remind yourself of these results.

a. First assume that ℓ passes through one of the infinite sides, say $\overrightarrow{P\Omega}$. Let R be the point of intersection on $\overrightarrow{P\Omega}$, and draw the segment $\overline{QR}$.

   (i) Draw this picture, including the points P, Q, Ω, and R, the sides of the omega-triangle, and the segment $\overline{QR}$. Do not draw ℓ yet. Notice that by assumption, ℓ must pass through some point X which is interior to the triangle, but you do not know whether X is in the interior of ΔPQR or of the new omega-triangle $QR\Omega$. (Why can't X be on the boundary?)

   (ii) Suppose X is in the interior of ΔPQR. Then what?

   (iii) Suppose X is in the interior of $QR\Omega$. Then what?

b. Now assume ℓ intersects $\overline{PQ}$ (but still does not pass through any vertices). Let R be the point of intersection, and find the limiting parallel $\overleftrightarrow{R\Omega}$. Show that any line through R and not through Ω (such as our line ℓ!) must intersect either $\overleftrightarrow{P\Omega}$ or $\overleftrightarrow{Q\Omega}$.

c. Take stock. Convince yourself that you've finished the proof.

Exercise 2: Angle of parallelism
Recall the definition of the angle of parallelism: We started with a line ℓ and a point P not on ℓ. We found a line m which was limiting parallel to ℓ through P; we drew the perpendicular line p from P to ℓ, and we measured the angle that it formed with m at P. This was the *angle of parallelism* of ℓ at P. We proved that it is always acute.

a. In this exercise, we will prove that the angle depends on only one thing: the distance h between P and the point Q where the perpendicular line p intersects the line ℓ.

   More precisely, let ℓ' be another line. Let Q' be a point on ℓ'; let p' be a line through Q' perpendicular to ℓ', and let P' be a point on p' such that the distance between P' and Q' is h. Let m' be limiting parallel to ℓ' at P'. Prove that the angle of parallelism in this case is congruent to the angle of parallelism of ℓ at P. (Use omega-triangle congruence.)

b. It follows that given any positive number h, we can find an angle $a(h)$, uniquely determined (up to congruence). This is called the *angle of parallelism* of h. We can also view $a(h)$ as a number between 0 and 90, the angle measure (e.g. in the Poincaré model).

c. Prove that if $h < h'$, then $a(h) > a(h')$ (i.e. the function a is *order-reversing*).

   Hint: start with a line ℓ, a point Q, and the perpendicular p to ℓ through Q. Draw two points P and P' on p corresponding to h and h', and draw the limiting parallels to ℓ at these points. Identify $a(h)$ and $a(h')$ in your picture. Use the Exterior Angle Theorem for Omega-Triangles.

d. Conclude that $a : \mathbb{R}_{>0} \to [0, 90)$ is an injective function: that is, given $h_1 \neq h_2$, prove that $a(h_1) \neq a(h_2)$.