MATH 402 Review for October 8–12

Topics: Euclidean isometries (5.1), reflections (5.2), composing reflections to get rotations and translations.
These were covered in lecture and in Worksheet 5. This material will also appear in Homework 6.

1. **Recall from last week:**
   (a) We defined transformations and isometries. You proved that isometries form a group.

2. **Things to know about isometries of the plane:**
   (a) We proved this big theorem that tells us that an isometry is bijective, sends lines to lines, and preserves angle measure.
   (b) We defined what it means for a set to be *invariant* under f and *fixed* by f.
   (c) We showed that if f is an isometry and S is its fixed point set, S can only be of a few different forms:
      i. $S = \varnothing$.
      ii. S is a point.
      iii. S is a line ℓ (and in this case f is a *reflection*, by definition).
      iv. S is everything (and in this case $f = \text{id}$).
   (d) In particular, to check whether an isometry is the identity, it is enough to check that it has three non-collinear fixed points. As a corollary, it is a homework problem to prove that two isometries which agree on three non-collinear points are equal.

3. **Things to know about reflections:**
   (a) Definitions: reflection, line of reflection.
   (b) Given two distinct points P and P' there is exactly one reflection r such that $r(P) = r(P')$.
   (c) Any isometry can be written as a composition of at most three reflections.

4. **Things to know about composing reflections:**
   (a) Definitions: translations, rotations.
   (b) $r_m \circ r_\ell \circ r_m = r_{r_m(\ell)}$. (Make sure you understand what this says; the notation can be confusing.)
   (c) A non-identity translation has no fixed points.
   (d) A non-identity rotation has exactly one fixed point.

Practice Questions

1. **Practice with reflections:**

   - Prove that $r^2 = \text{id}$.
   - Using this fact, and the definition of translations and rotations, explain why id is *both* a translation and a rotation. (Geometrically, it's "translation by zero" and "rotation by zero".) Is there a way to think of id as a reflection? (Either geometrically or from the definition.)[1]
   - Draw two congruent triangles. Find a sequence of reflections which takes one of them to the other. Find a different sequence. Observe that the decomposition of an isometry into a sequence of reflections is not unique.
   - On Wednesday, we proved the theorem mentioned in 3(b) on the previous page. We defined a function r, and we had to prove it was an isometry, i.e. that $CD = r(C)r(D)$. In class, we checked two cases: $C, D \in \ell$; and $C, D \notin \ell$ but on the same side of ℓ. Prove the remaining two cases: one of C, D is in ℓ; and C, D are on opposite sides of ℓ. Make sure you understand why these four cases cover all possibilities.

[1] Hint: no.