MATH 402 Review for September 4–7

Topics: 2.1 (proving things in Euclidean geometry) and 2.2 (proving things using Hilbert's axioms).;1.7 (experimenting with hyperbolic geometry).
These were covered in lecture, on Worksheet 1, and in the Project (1.7). This material will also appear in Homework 2.

1. **Recall from last week:**
   (a) Euclid had five axioms, the fifth of which is the **parallel postulate**. Know what they say.
   (b) The parallel postulate is equivalent to **Playfair's Postulate**. Know what it says.
   (c) When we use only the first four axioms, we call it **neutral geometry**.
   (d) When we replace the parallel postulate by the following statement, we obtain a consistent axiomatic system called **hyperbolic geometry**.
   Given a line ℓ and a point P not on ℓ, there are at least two lines through P parallel to ℓ.

2. **Key facts about parallel lines:**
   (a) The following statement holds in neutral geometry (i.e. we don't need the parallel postulate to prove it): *Suppose that a line n crosses two lines ℓ and m such that any of the following hold:*
   - a pair of alternate angles is congruent;
   - a pair of corresponding angles is congruent;
   - the sum of two interior angles on the same side of n is equal to $180°$.

   Then ℓ and m are parallel.
   (b) The converse to this statement is true in Euclidean geometry.

3. **Things to know about triangle congruence results:**
   (a) SAS is an axiom in Hilbert's formulation of Euclidean geometry.
   (b) The other triangle congruence results (SSS, ASA, AAS) are theorems, which can be proven from SAS and other axioms in the system. They are all true in neutral geometry.

4. **Things to know about Pasch's axiom:**
   (a) Know the statement of Pasch's axiom.
   (b) Know what it means for two points to be *on the same side* or *on opposite sides* of a line. Understand how this definition allows us to say that a line divides the plane into two separate sets (called *half-planes*).

5. **Some other theorems you should know:**
   (a) The Vertical Angle Theorem.
   (b) The Supplementary Angle Theorem.
   (c) The Exterior Angle Theorem.
   (d) The Isosceles Triangle Theorem.

Practice Questions

1. **Practice with exterior angles, alternate angles, corresponding angles**

   - Draw two lines ℓ and m, and draw a third line n passing through both of them at points A and B. Label a pair of alternate angles. Label a pair of corresponding angles. Label the interior angles.
   - Draw a triangle ΔABC. Extend one of the sides. Label the exterior angle, and label the two interior opposite angles. (Some people find this confusing, so try a few different orientations to make sure you're totally comfortable with it.) What does the exterior angle theorem tell you?

2. **Practice with parallel lines**
   Recall Euclid's Proposition 29 (Theorem 2.9 in the book). In class, we proved the second part of the proposition: if a line falls across two parallel lines, the resulting corresponding angles are conguent. We used Playfair's Postulate to do this.

   Prove the remaining two parts of the theorem: the resulting alternate angles are also congruent, and the sum of the interior angles on the same side of the line is equal to $180°$. For each statement, there are two approaches: either try to prove it directly using Playfair's Postulate (or the Parallel Postulate), or else use the fact that we already proved about the corresponding angles to deduce the desired result. Try either one of these approaches, or do both for extra practice.

3. **Practice with geometric constructions**
   Given a segment $\overline{AB}$, show how to construct the midpoint of the segment.

   Given a line ℓ and a point P not on ℓ, show how to construct the line through P perpendicular to ℓ.

   Given a line ℓ and a point P of ℓ, show how to construct the line through P perpendicular to ℓ.

   Try one of the following "games", which give you puzzles involving geometric constructions: https://kasperpeulen.github.io/ or https://www.euclidea.xyz/ (or just search for "Euclidean geometry games" to find other apps—they're all pretty similar). These are great practice to get comfortable with the techniques we use to prove theorems in Euclidean geometry.