MATH 402 Homework 3

Due Friday September 22, 2017

Exercise 1. This exercise will reinforce the ideas we have learned about inscribed angles, and show how they can be useful to prove other things. Recall that we studied the following theorem:

Theorem 1. *The measure of an angle inscribed in a circle is half that of its intercepted central angle.*

a. *[10 pts]* To prove this theorem, we let O be the centre of our circle c, and we denote our inscribed angle by $\angle APB$. We let Q be the second intersection point of the circle with the line $\overleftrightarrow{PO}$. We proved that $m\angle APB = \frac{1}{2} m\angle AOB$ in the case that A and B are on opposite sides of the line $\overleftrightarrow{PO}$. Finish the proof by showing that it is also true when
   (i) A and B are on the same sides of the line $\overleftrightarrow{PO}$.
   (ii) one of them, say A, is actually *on* the line $\overleftrightarrow{PO}$; that is, $A = Q$.

b. *[5 pts]* Now suppose that $ABCD$ is a quadrilateral inscribed in a circle. Prove that the angle at A and the angle at C are supplementary.

c. *[5 pts]* We can use this to complete the proof of the Law of Sines. Let ΔABC be a triangle inscribed in a circle σ of diameter d. We want to show that $\frac{a}{\sin \angle A} = d$, where a is the length of the side $\overline{BC}$. To prove this, we drew the diameter through B and the centre O of the circle, and we let D be the second intersection point of this diameter with the circle. We already proved the formula in the case that A and D are on the *same side* of the line $\overleftrightarrow{BC}$. Now prove it in the case that A and D are on *opposite sides.*

Exercise 2. This question is about vector geometry.

a. *[5 pts]* Let A and B be two distinct points. Show that the segment $\overline{AB}$ consists of all points C of the form $\vec{C} = \vec{A} + t(\vec{B} - \vec{A})$ where $t \in [0, 1]$.

b. *[5 pts]* Now show that the midpoint of the segment $\overline{AB}$ is the point M such that $\vec{M} = \frac{1}{2}(\vec{A} + \vec{B})$.

Exercise 3.

a. *[3 pts]* Solve exercise 2.6.5: show that the line passing through the centre of a circle and the midpoint of a chord (which is *not* a diameter is perpendicular to the chord.

b. *[3 pts]* Solve exercise 2.6.12: show that the line from the centre of a circle to an outside point bisects the angle made by the two tangents from that point to the circle. *Hint: use exercise 2.2.11.*

Exercise 4. This exercise is about Poincaré lines and the Poincaré distance formula. You may find it helpful to recall the following theorem, which you experimented with in Project 3.

Theorem 2. *Given two points P and Q inside a circle c, which are distinct from each other and from the centre O, there exists a unique circle or line through P and Q which is orthogonal to the circle c.*

a. *[4 pts]* Prove that two distinct Poincaré lines ℓ and ℓ' intersect at most once inside the Poincaré disk.

b. *[2 pts]* Recall that hyperbolic distance is defined by the formula

$$d_P(P,Q) = \left| \ln \left(\frac{(PS)}{(PR)} \frac{(QR)}{(QS)} \right) \right|.$$

Draw a picture showing P, Q, R, and S.

c. *[5 pts]* Show that $d_P(P,Q) = 0$ if and only if $P = Q$.

d. *[3 pts]* If $Q = O$ is the centre of the unit circle, simplify the formula for $d_P(P,Q)$.

Remember that in addition to the points assigned to each question, you will receive up to five further points for neatness and organization.