Regularisation Effect of Nonlinear Semigroups

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Daniel Hauer

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Motivation

Consider the following diffusion equation

$$\partial_t u - \mathrm{div}\,(a(x, \nabla u)\nabla u) + f(x,u) = 0 \quad \text{in } \Omega \times (0,\infty)$$

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$$u = 0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{(homog. Dirichlet)}$$

$$a(x,\nabla u)\cdot\vec{\nu} = 0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{(homog. Neumann)}$$

$$a(x,\nabla u)\cdot\vec{\nu} + \beta(x,u) = 0 \quad \text{on } \partial\Omega\times(0,\infty) \qquad \text{(homog. Robin)}$$

$$\lim_{|x|\to\infty} u(x,t) = 0 \quad \text{for } t>0 \qquad \text{(vanishing at } \infty)$$

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Here, $\Omega \subseteq \mathbb{R}^d$ open set, $\partial\Omega$ = bdry of Ω &
$f : \Omega \times \mathbb{R} \to \mathbb{R}$ Lipschitz, $f(x,0)=0$.

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and $a : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ is a Caratheodory function satisfying

▷ $|a(x,\xi)| \leq \alpha(x)|\xi|^{p-1} + h(x)$
$x\in\Omega,\ \xi\in\mathbb{R}^d$

▷ $a(x,\xi)\xi \geq \eta|\xi|^p$

▷ $(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2) > 0 \quad \xi_1 \neq \xi_2$

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$\varphi : \mathbb{R} \to \mathbb{R}$ is contininous & nondecreasing

Every solution $t \mapsto u(\cdot,t)$ of diffusion equation

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$$u(0) \in L^q \implies u(t) \in L^\infty \text{ for all } \underline{\underline{t>0}}$$

$$\& \quad \|u(t)\|_\infty \lesssim e^{\omega t \beta_q}\, t^{-\delta_q}\, \|u(0)\|_q^{\gamma_q}$$

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$$\boxed{\text{if } \varphi(s) = |s|^{m-1}s \text{ for } s\in\mathbb{R} \;\&\; m>0}$$

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$$\int_{\mathbb{R}^d} |f|^2 \ln |f| \, d\mu \le C \cdot \underbrace{\int_{\mathbb{R}^d} |\nabla f|^2 d\mu}_{= (-Af, f)_2} + \|f\|_2^2 \cdot \ln \|f\|_2$$

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2. When $r : [0,\infty) \to [p_0,\infty)$ is nondecreasing & C^1, then one shows by using the Log-Sobolev inequality that

$$s \mapsto y(s) := \log \lVert u(s) \rVert_{r(s)}$$

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$\Longrightarrow$ L^q-L^r-regularising effect of $u(0) \mapsto u(t)$, $t>0$.

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On builds a "one parameter family" of
(Log-) Sobolev inequalities.

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For $q \geq p$

$$\langle -\Delta_p u, |u|^{q-p} u \rangle = (q-p+1) \int_\Omega |\nabla u|^p \, |u|^{q-p} dx$$

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$$\geq C_{qp} \left(\int_\Omega \left| |u|^{\frac{q-p}{p}} u \right|^{p^*} dx \right)^{\frac{p}{p^*}}$$

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$$\partial_t u - (-\Delta_p)^s u + \beta(u) + f(x,u) \ni 0 \quad \text{in } \Omega \times (0,\infty)$$

where $(-\Delta_p)^s u(x) := \text{P.V.} \int_{\Omega} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{d+sp}}\,dy$

for $0 < s < 1$

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$$(1)\begin{cases} \partial_t u - a(x,\nabla u)\cdot\vec{\nu} + \beta(u) + f(x,u) \ni 0 & \text{on } \partial\Omega\times(0,\infty) \\ -\mathrm{div}(a(x,\nabla u)\nabla u) = 0 & \text{in } \Omega\times(0,\infty) \end{cases}$$

What is known about the regularisation effect of non local diffussion equations?

If $P : \partial\Omega \ni \varphi \longmapsto P\varphi := u$ unique solution of

$$(DP_\varphi) \quad \begin{cases} -\operatorname{div}(a(x,\nabla u)\nabla u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

then $\Lambda : \partial\Omega \ni \varphi \longmapsto a(x, P\varphi) \cdot \vec{\nu}_{|\partial\Omega}$ is the

Dirichlet-to-Neumann operator

associated with $-\operatorname{div}(a(x,\nabla u)\nabla u) = 0$

What is known about the regularisation effect of non local diffussion equations?

Note, (1) is equivalent to

(2) $\quad \partial_t \varphi + a(x, \nabla\varphi)\cdot\vec{\nu} + \beta(\varphi) + f(x,\varphi) \ni 0$ on $\partial\Omega\times(0,\infty)$

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A an operator on X with domain $D(A)$.

For given $u_0 \in X$ let $u : [0,\infty) \to X$ be a solution of
(in some sense)

$$(\mathcal{CP}) \quad \begin{cases} \frac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t > 0, \\ u(0) = u_0. \end{cases}$$

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(Accretive) for every $\lambda > 0$ the resolvent oper. $J_\lambda := (1+\lambda A)^{-1}$ is X-contractive,

$$\|J_\lambda u - J_\lambda \hat{u}\|_X \le \|u - \hat{u}\|_X \quad \forall u, \hat{u} \in Rg(1+\lambda A)$$

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Then by Crandall–Liggett '71,

$$(CP)\quad \begin{cases} \frac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t>0, \\ u(0) = u_0 \in \overline{D(A)} \end{cases}$$

is well-posed in the sense of mild solutions.

The framework of this talk

Now, for given $u_0 \in X$ a function $u \in C([0,\infty);X)$ is called a mild solution of (EP) if for every $T>0$, $\varepsilon>0$ and partition $0=t_0<t_1<\dots<t_N=T$ of $[0,T]$ s.th. $t_i - t_{i-1} < \varepsilon$ $\forall i=1,\dots,N$

there is a piecewise constant function $u_{\varepsilon,N}(t) := \sum_{i=1}^{N} u_{\varepsilon,i} \mathbb{1}_{(t_{i-1},t_i]}(t) + u_0$,

where $u_{\varepsilon,i}$ on $(t_{i-1},t_i]$ solves "recursively"

the finite difference equation $u_{\varepsilon,i} + (t_i - t_{i-1}) A u_{\varepsilon,i} \ni u_{\varepsilon,i-1}$ $\forall i=1,\dots,N$

such that $\sup_{t\in[0,T]} \|u_{\varepsilon,N}(t) - u(t)\|_X \le \varepsilon$.

The framework of this talk

By the celebrated Crandall–Liggett Thm (1971), if A is m-accretive in X, then for every $u_0 \in \overline{D(A)}$ there is a unique mild solution of (EP) and the solution is given by the exponential formula

$$u(t) := \lim_{n\to\infty} \left(1+\frac{t}{n}A\right)^{-n} u_0$$ unif. on comp. subintv. of $[0,\infty)$.

The framework of this talk

For given $u_0 \in X$, we call a function $u \in C([0,\infty);X)$ a strong solution of (EP) if $u \in W^{1,1}_{loc}((0,\infty);X)$, $u(0) = u_0$ in X and for a.e. $t > 0$ one has

$$u(t) \in D(A) \quad \& \; -\frac{du}{dt}(t) \in Au(t).$$

The framework of this talk

For given $u_0 \in \overline{D(A)}$, let $u : [0,\infty) \to X$ be the the mild solution of (EP), then set

$$T_t u_0 := u(t) \quad \forall t \geq 0.$$

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Then the family $\{T_t\}_{t\geq 0}$ of mappings $T_t : \overline{D(A)} \to \overline{D(A)}$ satisfies:

- $T_t \circ T_s = T_{t+s} \quad \forall t, s \geq 0$ (semi-group)
- $\lim_{t \to 0+} \|T_t u - u\|_X = 0 \quad \forall u \in \overline{D(A)}$ (Strong cont.)
- $\|T_t u - T_t \hat{u}\|_X \leq \|u - \hat{u}\|_X \quad \forall u, \hat{u} \in \overline{D(A)}$ (X-contractive)

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The abstract framework

$$(\mathrm{EP})\quad \begin{cases} \frac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t>0, \\ u(0) = u_0 \in \overline{D(A)} \end{cases}$$

allows to establish existence, uniqueness & cont. dependence on u_0 of many important parabolic bdry-value problems:

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▷ $X = L^2(\Omega)$: $A = -\mathrm{div}(a(x,\nabla u))$

$\Omega \subseteq \mathbb{R}^d$ domain or $A = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$

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allows to establish existens, uniqueness & cont. dependence on u_0 of many important parabolic bdry-value problems:

▷ $X = L^1(\Omega)$: $\quad A = -\Delta_p^D u^m \quad , u^m := |u|^{m-1}u$, $m>1$

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Ex: $-\Delta u^m$ or $-\Delta_p u^m$.

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$$A \text{ completely accretive} \implies A \text{ T-accretive in } L^q(\Sigma,\mu) \text{ for } 1 \le q < \infty.$$

▷ If $X \subseteq M(\Sigma,\mu)$ is a Banach lattice, then A is called T-accretive if

$$\|[u-\hat{u}]^+\|_X \le \|[u-\hat{u}+\lambda(v-\hat{v})]^+\|_X \quad \forall \lambda > 0 \;\&\; \forall (u,v), (\hat{u},\hat{v}) \in A.$$


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Main subject of this talk

To investigate the "regularising effect":

For given $u \in L^q(\Sigma,\mu)$ for some $q \geq 1$,

$$T_t u \in L^r(\Sigma,\mu) \quad \forall t>0 \;\&\; \text{some } q<r\leq\infty$$

+ uniform bdds/decay rates as $t \to \infty$.

Theorem [Coulhon, H. '15] Sobolev ⇒ L^2-L^r-reg

Let A be m-completely accretive in $L^2(\Sigma,\mu)$ with dense domain & $0 \in A0$.

If there are $2 \leq r \leq \infty$, $\sigma > 0$ & $c > 0$ s.th.

(Sob) $$\|u-\hat{u}\|_r^\sigma \leq c \cdot \langle u-\hat{u}, v-\hat{v}\rangle_2$$

for all $(u,v), (\hat{u},\hat{v}) \in A$, then $\{T_t\}_{t\geq 0} \sim -A$ satisfies

$$\|T_t u - T_t \hat{u}\|_r \leq \left(\frac{c}{2}\right)^{\frac{1}{\sigma}} t^{-\frac{1}{\sigma}} \|u-\hat{u}\|_2^{\frac{2}{\sigma}} \quad \forall u, \hat{u} \in L^2.$$

Proof of Theorem 1 :

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$$\|u-\hat{u}\|_q^q \geq \|u-\hat{u}\|_q^q - \|T_t u - T_t \hat{u}\|_q^q$$

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$$= (-q) \int_0^t \left\langle (u-\hat{u})_q, \frac{d}{ds} T_s u - \frac{d}{ds} T_s \hat{u} \right\rangle_{q', q} ds$$

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$$= - \int_0^t \frac{d}{ds} \|T_s u - T_s \hat{u}\|_q^q \, ds$$

$$= (-q) \int_0^t \langle (u-\hat{u})_q , \tfrac{d}{ds} T_s u - \tfrac{d}{ds} T_s \hat{u} \rangle_{q',q} \, ds$$

$$= q \int_0^t \langle (u-\hat{u})_q , A T_s u - A T_s \hat{u} \rangle_{q',q} \, ds$$

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$$\|u-\hat{u}\|_q^q \geq \|u-\hat{u}\|_q^q - \|T_t u - T_t \hat{u}\|_q^q$$

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1* Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^2 - L^r$-reg

Let A be m-accretive in $L^1(\Sigma, \mu)$ with non-increasing resolvent in all L^q & dense domain.

If there are $2 \leq r \leq \infty$, $\sigma > 0$ & $c > 0$ s.th.

(Sob) $$\|u\|_r^\sigma \leq c \, \langle u, v \rangle_2$$

for all $(u,v) \in A \cap (L^2 \times L^2)$, then $\{T_t\}_{t \geq 0} \sim -A$ satisfies

$$\|T_t u\|_r \leq \left(\frac{c}{2}\right)^{\frac{1}{\sigma}} t^{-\frac{1}{\sigma}} \|u\|_2^{\frac{2}{\sigma}} \qquad \forall u \in L^2.$$

THE UNIVERSITY OF SYDNEY

Daniel Hauer

2 Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^q$-L^∞-reg.

Let A be m-completely accretive in $L^2(\Sigma,\mu)$ with dense domain & $0 \in A0$.

If there are $2 < r < \infty$, $\sigma > 0$ & $C > 0$ s.th. $\frac{r}{\sigma} > 1$ &

(Sob) $\quad \|u-\hat{u}\|_r^\sigma \le C \cdot \langle u-\hat{u}, v-\hat{v}\rangle_2$

for all $(u,v), (\hat{u},\hat{v}) \in A$, then $\{T_t\}_{t\ge 0} \sim -A$ satisfies

$$\|T_t u - T_t \hat{u}\|_\infty \lesssim t^{-\delta} \, \|u-\hat{u}\|^{\gamma}_{\frac{r}{\sigma} m_0} \quad \forall u, \hat{u} \in L^{\frac{r}{\sigma} m_0},$$

for any $m_0 > \sigma$ satisfying $\left(\frac{r}{\sigma}-1\right) m_0 + \sigma - 2 > 0$

with

$$\delta = \frac{1}{\left(\frac{r}{\sigma}-1\right) m_0 + \sigma - 2}, \quad \gamma = \frac{\left(\frac{r}{\sigma}-1\right) m_0}{\left(\frac{r}{\sigma}-1\right) m_0 + \sigma - 2}.$$

3 Theorem [Coulhon, H. '15] Extrapolation towards L^1

For $1 \le q < r \le \infty$, let $\{T_t\}_{t\ge 0}$ be L^1-contractive semigroup on $L^1 \cap L^r(\Sigma,\mu)$, $T_t 0 \equiv 0$ & suppose, there are $\delta, \gamma > 0$ s.th.

$$\|T_t u - T_t \hat{u}\|_r \le C\, t^{-\delta}\, \|u-\hat{u}\|_q^{\gamma} \quad \forall t>0 \ \& \ \forall u,\hat{u} \in L^1 \cap L^r$$

For $\Theta_r := \frac{r-q}{q(r-1)} > 0$ if $r<\infty$ & $\Theta_\infty := \frac{1}{q}$ if $r=\infty$

assume that $\gamma(1-\Theta_r) < 1$.

Then

$$\|T_t u - T_t \hat{u}\|_r \le (2^\delta C)^{1/\Theta}\, t^{-\frac{\delta}{\Theta}}\, \|u-\hat{u}\|_1^{\gamma \frac{\Theta_r}{\Theta}} \quad \forall u,\hat{u} \in L^1(\Sigma,\mu)$$

with $\Theta := 1-\gamma(1-\Theta)$.

Applications

Applications

1. Total variational flow in $\mathbb{R}^d$

$$A := \left\{ (u,v) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} u \in BV(\mathbb{R}^N) \;\&\; \exists z \in L^\infty(\mathbb{R}^d, \mathbb{R}^d), \|z\|_\infty \leq 1, \\ v = -\operatorname{div} z \text{ in } \mathcal{D}' \;\&\; (z, Du) = |Du| \end{array} \right\}$$

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Sobolev in.

$$\|u\|_{L^{\frac{d}{d-1}}} \leq C\, |Du|(\mathbb{R}^d)$$

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1. Total variational flow in $\mathbb{R}^d$

$\Rightarrow \forall (u,v) \in A, \quad \langle u, v \rangle_2 = \int_{\mathbb{R}^d} -\mathrm{div}(z) \cdot u \, dx$

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Green's formula (Anzelotti)

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$$\overset{\text{Green's formula (Anzelotti)}}{=} \int_{\mathbb{R}^d} (z, Du)$$

$$= \int_{\mathbb{R}^d} |Du|$$

$$\geq C^{-1} \|u\|_{L^{\frac{d}{d-1}}}$$

Applications

1. Total variational flow in $\mathbb{R}^d$

Thm 1* $\Longrightarrow$ $\|T_t u\|_{\frac{d}{d-1}} \leqslant \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t > 0$
$\& \; \forall u \in L^2(\mathbb{R}^d)$

Applications

1. Total variational flow in $\mathbb{R}^d$

Thm 1* $\Longrightarrow$

$$\|T_t u\|_{\frac{d}{d-1}} \leqslant \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t>0$$
$$\& \ \forall u \in L^2(\mathbb{R}^d)$$

Since $\frac{d}{d-1} > 1$,

Thm 2* $\Longrightarrow$

$$\|T_t u\|_\infty \lesssim t^{-\frac{1}{(\frac{d}{d-1}-1)m_0-1}} \|u\|_{\frac{dm_0}{d-1}}^{\frac{(\frac{d}{d-1}-1)m_0}{(\frac{d}{d-1}-1)m_0-1}} \quad \forall u \in L^{\frac{dm_0}{d-1}}$$

for any $m_0 > 1$ s.th. $\left(\frac{d}{d-1}-1\right)m_0 - 1 > 0$

Thank You!