Equadiff 2015

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A simplified approach to the regularising effect of nonlinear semigroups

A joint work with Prof Thierry Coulhon (PSL/ANU)

Daniel Hauer

THE UNIVERSITY OF SYDNEY

09.07.2015

The framework of this talk

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$X = L^q(\Sigma,\mu)$ for some $1 \le q \le \infty$,
A an operator on X with domain $D(A)$.

For given $u_0 \in X$ let $u : [0,\infty) \to X$ be a solution of
(in some sense)

$$(\mathcal{CP}) \quad \begin{cases} \frac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t > 0, \\ u(0) = u_0. \end{cases}$$

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(Accretive) for every $\lambda > 0$ the resolvent oper. $J_\lambda := (1+\lambda A)^{-1}$ is X-contractive,

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Then by Crandall–Liggett '71,

$$(\mathrm{CP}) \begin{cases} \frac{du}{dt}(t) + Au(t) \ni 0 & \text{for } t > 0, \\ u(0) = u_0 \in \overline{D(A)} \end{cases}$$

is well-posed in the sense of mild solutions.

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For given $u_0 \in \overline{D(A)}$, let $u : [0,\infty) \to X$ be the the mild solution of (EP), then set

$$T_t u_0 := u(t) \quad \forall t \geq 0.$$

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Then the family $\{T_t\}_{t\geq 0}$ of mappings $T_t : \overline{D(A)} \to \overline{D(A)}$ satisfies:

- ▷ $T_t \circ T_s = T_{t+s} \quad \forall t, s \geq 0$ (semi-group)
- ▷ $\lim_{t\to 0+} \|T_t u - u\|_X = 0 \quad \forall u \in \overline{D(A)}$ (Strong cont.)
- ▷ $\|T_t u - T_t \hat{u}\|_X \leq \|u - \hat{u}\|_X \quad \forall u, \hat{u} \in \overline{D(A)}$ (X-contractive)

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The abstract framework

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allows to establish existence, uniqueness & cont. dependence on u_0 of many important parabolic bdry-value problems:

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▷ $X = L^2(\Omega)$: $\quad A = -\mathrm{div}(a(x,\nabla u))$

$\Omega \subseteq \mathbb{R}^d$ domain $\quad$ or $\quad A = -\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$

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To investigate the "regularising effect":

For given $u \in L^q(\Sigma, \mu)$ for some $q \geq 1$,

$$T_t u \in L^r(\Sigma, \mu) \quad \forall t > 0 \ \& \text{ some } q < r \leq \infty$$

+ uniform bdds/decay rates as $t \to \infty$.

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Outline of this talk

1. Log-Sobolev is not always the direct & simplest way
2. Other results to this subject
3. Our motivation / AIM
4. Main theorems
5. Applications

① Log-Sobolev is not always the direct & simplest way

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satisfies a differential inequality.

$\Longrightarrow$ L^q-L^r-regularising effect of $u(0)\mapsto u(t)$, $t>0$.

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Let $\{T_t\}_{t\geq 0}$ be a semigroup of bounded linear operators T_t acting on $L^q(\Sigma,\mu)$ for all $1\leq q\leq\infty$, of a measure space (Σ,μ), with infinitesimal gen. $-A$.

$\{T_t\}_{t\geq 0}$ is hypercontractive, i.e., for some (all) $1<q<r<\infty$ $\exists t=t(q,r)>0$ s.th. T_t maps $L^q(\Sigma,\mu)$ to $L^r(\Sigma,\mu)$	iff Log-Sobolev holds for the generator $-A$ of $\{T_t\}_{t\geq 0}$.

① Log-Sobolev is not always the direct & simplest way

▷ Log-Sobolev inequality: Take $d\mu := (2\pi)^{-d/2} e^{-\frac{x^2}{2}} dx$

$$\int_{\mathbb{R}^d} |f|^2 \ln |f| \, d\mu \le C \cdot \int_{\mathbb{R}^d} |\nabla f|^2 d\mu + \|f\|_2^2 \cdot \ln \|f\|_2$$

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The semigroup $\{T_t\}_{t\geq 0} \sim -A$ is the Ornstein-Uhlenbeck semigroup.

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In the 80's, the focus shifted towards a the stronger property called ultracontractivity:

for all $t>0$, T_t maps $L^1(\Sigma,\mu)$ to $L^\infty(\Sigma,\mu)$

& of particular interest:

(UC) $\quad \|T_t\|_{\mathcal{L}(L^1,L^\infty)} \lesssim t^{-d/2}$ for all $t>0$.

① Log-Sobolev is not always the direct & simplest way

In 1985, Varopoulos proved:

For a given (Σ, μ), one has:

$\{T_t\}_{t\geq 0}$ satisfies (UC) $\iff$ The generator $-A$ of $\{T_t\}_{t\geq 0}$ satisfies a d-dim Sobolev ineq

$$\|f\|^p_{\frac{pd}{d-2}} \leq C \cdot \langle Af, f_p \rangle$$

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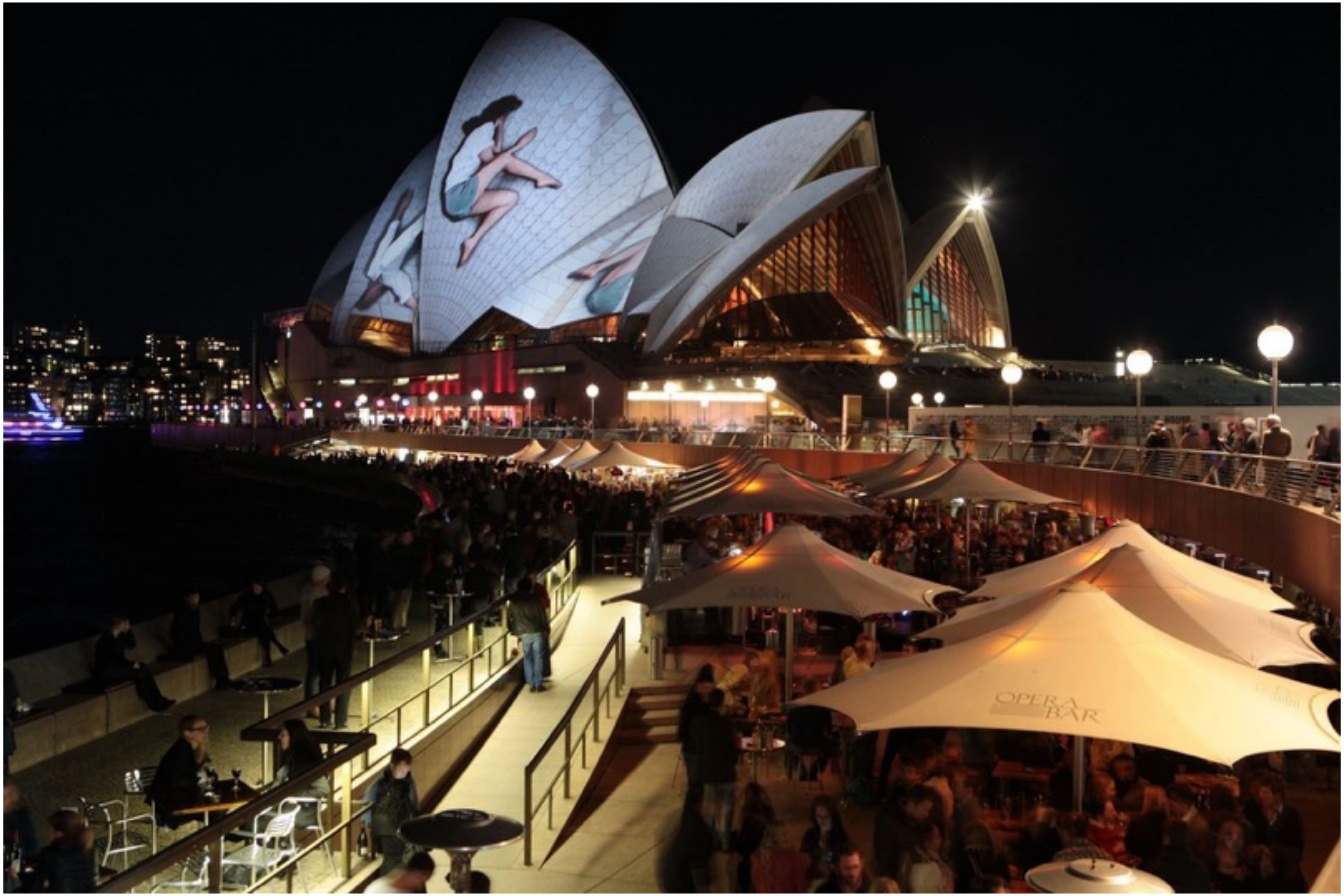
by Varopoulos $\Longleftrightarrow$ The generator $-A$ of the O-U semigroup does not satisfy a d-dim. Sobolev ineq.

In other words, for the measure $d\mu = (2\pi)^{-d/2} e^{-x^2/2} dx$ a d-dim Sobolev inequality is not valid.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Many others but for more specific situations.

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$$\langle -\Delta_p u, |u|^{q-p} u \rangle = (q-p+1) \int_\Omega |\nabla u|^p |u|^{q-p} dx$$

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$$\geq C_{qp} \left(\int_\Omega \left| |u|^{\frac{q-p}{p}} u \right|^{p^*} dx \right)^{\frac{p}{p^*}}$$

② Other results to this subject

Main tool in <u>all</u> these approaches
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On builds a "one parameter family" of
(Log-) Sobolev inequalities.

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- ▷ the construction of one-parameter families of (Log-) Sobolev ineq.
- ▷ too much regularity.

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To develope a unified machinery,
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▷ completely accretive operators A,
i.e., $\forall \lambda > 0$ the resolvent J_λ is L^q-contractive
+ order preserving $\forall\, 1 \le q \le \infty$

③ Our motivation / AIM

To develope a unified machinery, which concerns

- ▷ completely accretive operators A, i.e., $\forall \lambda > 0$ the resolvent J_λ is L^q-contractive $\forall\, 1 \le q \le \infty$ + order preserving
- ▷ accretive operators A non-increasing resolvent J_λ in L^q $\forall\, 1 \le q \le \infty$ (i.e. $\|J_\lambda u\|_q \le \|u\|_q$)

④ Main theorems

Theorem [Coulhon, H. '15] Sobolev ⇒ L^2–L^r-reg

Let A be m-completely accretive in $L^2(\Sigma,\mu)$ with dense domain & $0 \in A0$.

If there are $2 \le r \le \infty$, $\sigma > 0$ & $c > 0$ s.th.

(Sob) $$\|u-\hat{u}\|_r^\sigma \le c\,\langle u-\hat{u}, v-\hat{v}\rangle_2$$

for all $(u,v), (\hat{u},\hat{v}) \in A$, then $\{T_t\}_{t\ge 0} \sim -A$ satisfies

$$\|T_t u - T_t \hat{u}\|_r \le \left(\frac{c}{2}\right)^{\frac{1}{\sigma}} t^{-\frac{1}{\sigma}} \|u-\hat{u}\|_2^{\frac{2}{\sigma}} \quad \forall u, \hat{u} \in L^2.$$

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1* Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^2$-L^r-reg

Let A be m-accretive in $L^1(\Sigma,\mu)$ with non-increasing resolvent in all L^q & dense domain.

If there are $2 \leq r \leq \infty$, $\sigma > 0$ & $c > 0$ s.th.

$$\text{(Sob)} \qquad \|u\|_r^\sigma \leq c \cdot \langle u, v\rangle_2$$

for all $(u,v) \in A \cap (L^2 \times L^2)$, then $\{T_t\}_{t\geq 0} \sim -A$ satisfies

$$\|T_t u\|_r \leq \left(\frac{c}{2}\right)^{\frac{1}{\sigma}} t^{-\frac{1}{\sigma}} \|u\|_2^{\frac{2}{\sigma}} \qquad \forall u \in L^2.$$

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④ Main theorems

2 Theorem [Coulhon, H. '15] Sobolev ⇒ L^q–L^∞-reg.

Let A be m-completely accretive in $L^2(\Sigma,\mu)$ with dense domain & $0\in A0$.

If there are $2<r<\infty$, $\sigma>0$ & $c>0$ s.th. $\frac{r}{\sigma}>1$ &

(Sob) $\quad \|u-\hat{u}\|_r^\sigma \le c\,\langle u-\hat{u}, v-\hat{v}\rangle_2$

for all $(u,v),(\hat{u},\hat{v})\in A$, then $\{T_t\}_{t\ge 0}\sim -A$ satisfies

$$\|T_t u - T_t\hat{u}\|_\infty \lesssim t^{-\delta}\,\|u-\hat{u}\|^{\gamma}_{\frac{r}{\sigma}m_0}\quad \forall u,\hat{u}\in L^{\frac{r}{\sigma}m_0},$$

for any $m_0>\sigma$ satisfying $\left(\frac{r}{\sigma}-1\right)m_0+\sigma-2>0$

with

$$\delta=\frac{1}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-2},\quad \gamma=\frac{\left(\frac{r}{\sigma}-1\right)m_0}{\left(\frac{r}{\sigma}-1\right)m_0+\sigma-2}.$$

④ Main theorems

2* Theorem [Coulhon, H. '15] Sobolev $\Rightarrow L^q$–L^∞-reg.

Let A be m-accretive in $L^1(\Sigma,\mu)$ with non-incr. resolvent J_λ in L^q, dense domain & $T_t \in \mathcal{C}(L^\infty, L^\infty)$ $\forall t \geq 0$.

If there are $2 < r < \infty$, $\sigma > 0$ & $c > 0$ s.th. $\frac{r}{\sigma} > 1$ &

(Sob) $$\|u\|_r^\sigma \leq C \langle u, v \rangle_2$$

for all $(u,v) \in A \cap (L^2 \times L^2)$, then $\{T_t\}_{t\geq 0} \sim -A$ satisfies

$$\|T_t u\|_\infty \lesssim t^{-\delta} \, \|u\|_{\frac{r}{\sigma} m_0}^{\gamma} \quad \forall u \in L^{\frac{r}{\sigma} m_0},$$

for any $m_0 > \sigma$ satisfying $\left(\frac{r}{\sigma} - 1\right) m_0 + \sigma - 2 > 0$ with

$$\delta = \frac{1}{\left(\frac{r}{\sigma} - 1\right) m_0 + \sigma - 2}, \quad \gamma = \frac{\left(\frac{r}{\sigma} - 1\right) m_0}{\left(\frac{r}{\sigma} - 1\right) m_0 + \sigma - 2}.$$

④ Main theorems

3 Theorem [Coulhon, H. '15] Extrapolation towards L^1

For $1 \le q < r \le \infty$, let $\{T_t\}_{t\ge 0}$ be L^1-contractive semigroup on $L^1 \cap L^r(\Sigma,\mu)$, $T_t 0 \equiv 0$ & suppose, there are $\delta, \gamma > 0$ s.th.

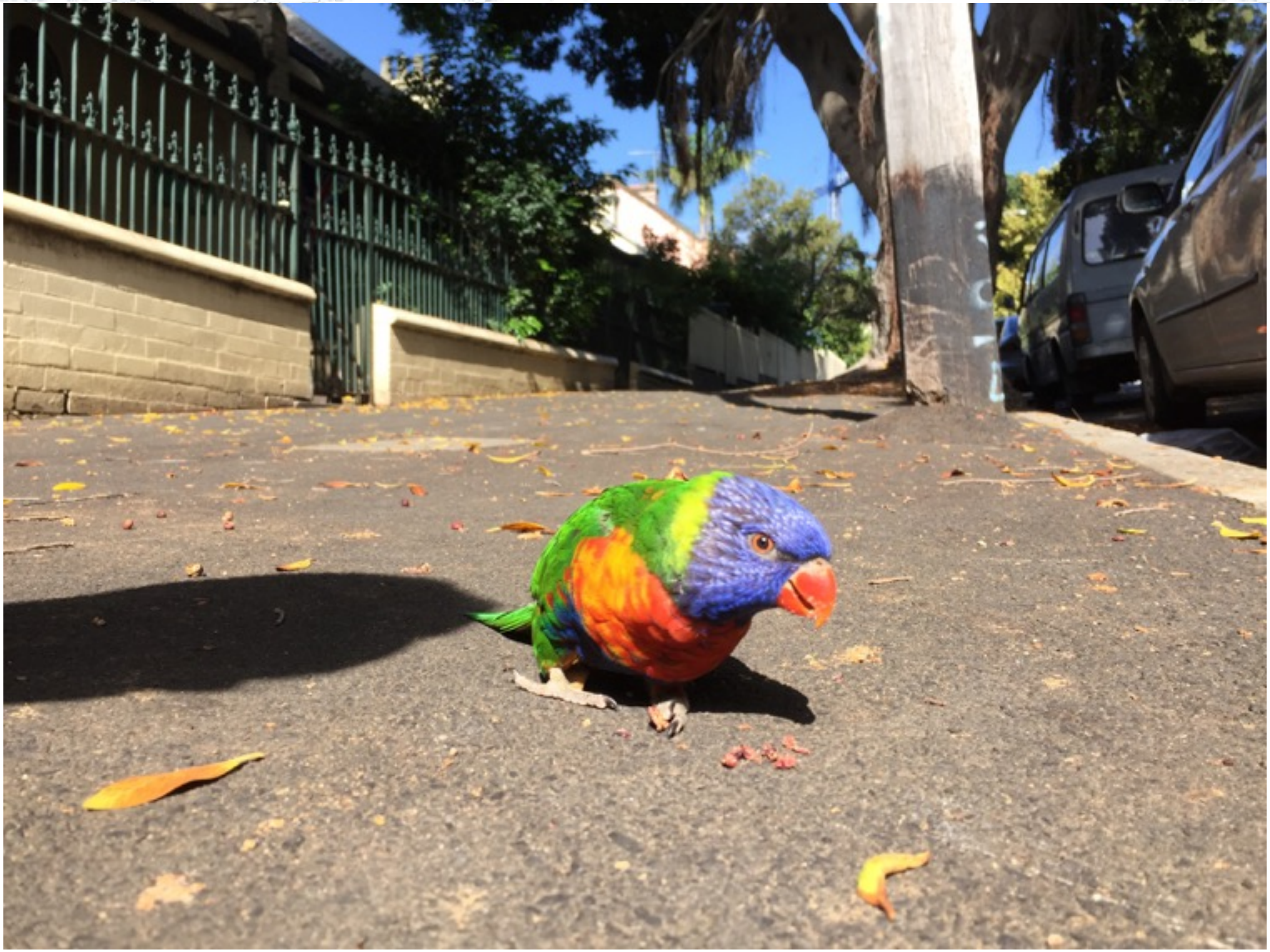
$$\|T_t u - T_t \hat{u}\|_r \le C\, t^{-\delta} \|u-\hat{u}\|_q^{\gamma} \quad \forall t>0 \ \& \ \forall u, \hat{u} \in L^1 \cap L^r$$

For $\Theta_r := \frac{r-q}{q(r-1)} > 0$ if $r < \infty$ & $\Theta_\infty := \frac{1}{q}$ if $r = \infty$

assume that $\gamma(1-\Theta_r) < 1$.

Then
$$\|T_t u - T_t \hat{u}\|_r \le (2^\delta C)^{1/\Theta}\, t^{-\frac{\delta}{\Theta}} \|u-\hat{u}\|_1^{\gamma \frac{\Theta_r}{\Theta}} \quad \forall u, \hat{u} \in L^1(\Sigma,\mu)$$

with $\Theta := 1 - \gamma(1-\Theta)$.



⑤ Applications

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1. Total variational flow in $\mathbb{R}^d$

$$A := \left\{ (u,v) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \,\middle|\, \begin{array}{l} u \in BV(\mathbb{R}^N) \ \& \ \exists z \in L^\infty(\mathbb{R}^d,\mathbb{R}^d),\ \|z\|_\infty \le 1, \\ v = -\operatorname{div} z \text{ in } \mathcal{D}' \ \& \ (z, Du) = |Du| \end{array} \right\}$$

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$\Rightarrow$ A is m-completely accretive in $L^2(\mathbb{R}^d)$ & $0 \in A0$.

Sobolev ineq.

$$\|u\|_{L^{\frac{d}{d-1}}} \leq C\, |Du|(\mathbb{R}^d)$$

⑤ Applications

1. Total variational flow in $\mathbb{R}^d$

$$\Rightarrow \forall (u,v) \in A, \quad \langle u, v \rangle_2 = \int_{\mathbb{R}^d} -\mathrm{div}(z) \cdot u \, dx$$

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$$\geq C^{-1} \|u\|_{L^{\frac{d}{d-1}}}$$

⑤ Applications

1. Total variational flow in $\mathbb{R}^d$

Thm 1* $\Longrightarrow$

$$\|T_t u\|_{\frac{d}{d-1}} \leq \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t > 0$$
$$\& \ \forall u \in L^2(\mathbb{R}^d)$$

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$$\overset{\text{Thm 1}^*}{\Longrightarrow} \quad \|T_t u\|_{\frac{d}{d-1}} \leq \frac{c}{2}\, t^{-1} \|u\|_2^2 \quad \forall t>0 \ \& \ \forall u \in L^2(\mathbb{R}^d)$$

Since $d/_{d-1} > 1$,

$$\overset{\text{Thm 2}^*}{\Longrightarrow} \quad \|T_t u\|_\infty \lesssim t^{-\frac{1}{(d/(d-1)-1)m_0-1}} \|u\|_{\frac{dm_0}{d-1}}^{\frac{(\frac{d}{d-1}-1)m_0}{(d/(d-1)-1)m_0-1}} \quad \forall u \in L^{\frac{dm_0}{d-1}}$$

for any $m_0>1$ s.th. $(\frac{d}{d-1}-1)m_0-1>0$

Thank You!