Symbolic dynamics of the reduced planar 3-body problem

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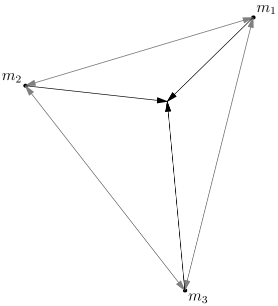
57th Annual Meeting of the Australian Mathematical Society

Planar 3-body problem

The problem of three massive particles moving under mutual Newtonian gravitation in 2D. Force of mass l on mass k is

$$F_{lk} = \frac{G m_l m_k (X_l - X_k)}{|X_l - X_k|^3}.$$

Each mass exerts a force on every other, so all attraction is to the common centre of mass.



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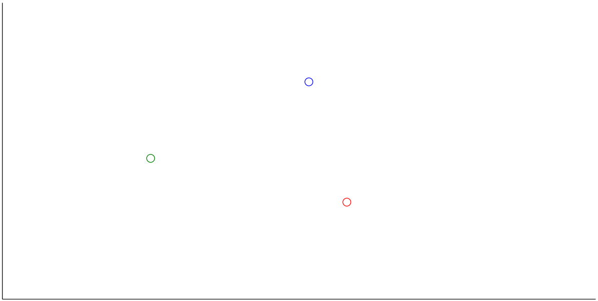

Introduction

Aim: To to develop a set of tools to study periodic and relatively periodic motions of the planar 3-body problem with vanishing angular momentum.

- Symmetry reduction
- Regularisation of binary collisions
- Symbolic dynamics
- Classification by discrete symmetries

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3-body Hamiltonian

Choosing units such that $G = 1$, the 3-body Hamiltonian is

$$H = T(\boldsymbol{P}) + U(\boldsymbol{X}), \text{ where}$$

$$T = \sum_{j=1}^{3} \frac{P_j^2}{2m_j} \text{ and}$$

$$U = -\frac{m_2 m_3}{|X_2 - X_3|} - \frac{m_3 m_1}{|X_3 - X_1|} - \frac{m_1 m_2}{|X_1 - X_2|},$$

in which $\boldsymbol{X}, \boldsymbol{P} \in \mathbb{C}^3$ with components indexed $j = 1, 2, 3$ are, respectively, coordinates and momenta of body j and $m_j \in \mathbb{R}^+$ is its mass.

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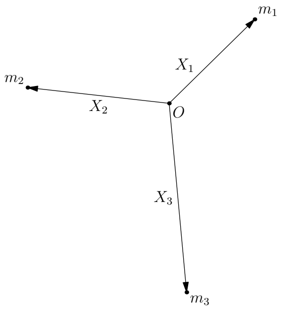
Symmetries 1

Can remove continuous symmetries. E.g. fix centre of mass at the origin:



and extract angular momentum to separate “shape dynamics” from “rotation dynamics”.

Symmetries 2

Discrete symmetries are:

- time reversal - τ
- spatial reflection - ρ
- permutation of labels - σ_j (swap k, l, preserve j), c (cycle).

In all, forming a group of order 24 under composition.
This group is $C_2 \times C_2 \times S_3$.

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- E.g. angular momentum $L = \sum \bar{X}_j P_j$ is a constant of motion (but you wouldn’t realise it from that expression right away!).
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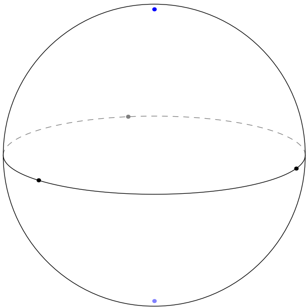
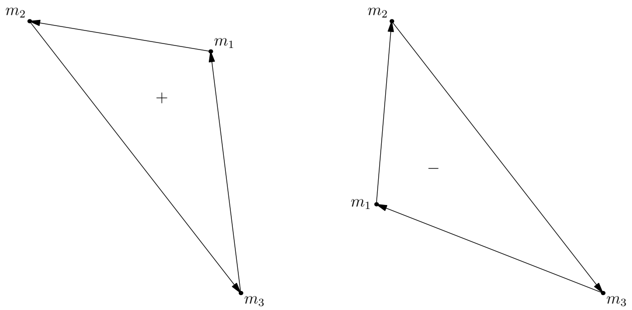
An obscure object



The shape sphere 1

Those 3 coordinates from before represent two angles and some length scale. This describes the shape space; each point is an oriented triangle.
Oriented triangles are triangles where the ordering of vertices is important:



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- Length scale is moment of inertia: $I = \sum m_j X_j$, the radial component of shape space.
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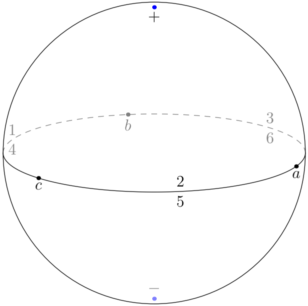

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- Equator is all collinear configurations.
- Three points on the equator represent binary collisions.

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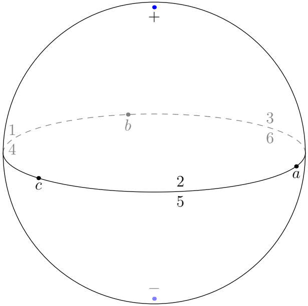

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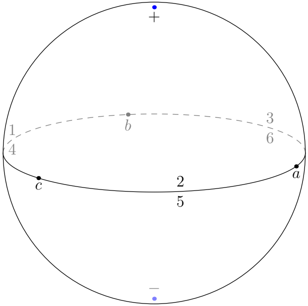

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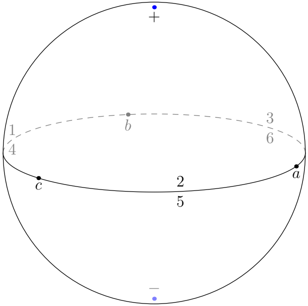

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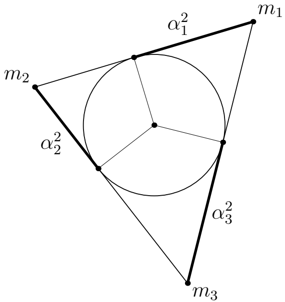
Regularising coordinates 1

Why regularise?

Even without collisions, it makes close encounters nice in numerical treatment.

Following [1], two steps to regularisation: first, make more space...

Choose new coordinates α_j s.t. $|X_l - X_k| = \alpha_k^2 + \alpha_l^2$.



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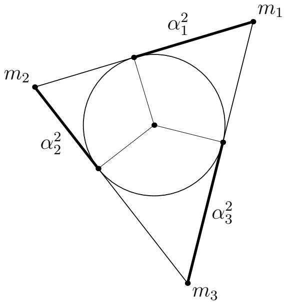

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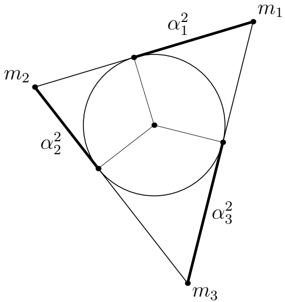

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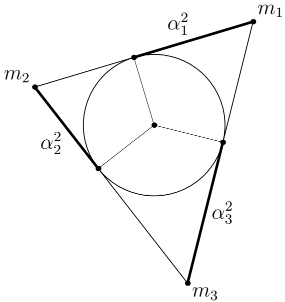

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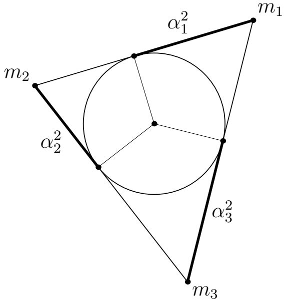
Canonical momenta π_j are introduced, conjugate to α_j.



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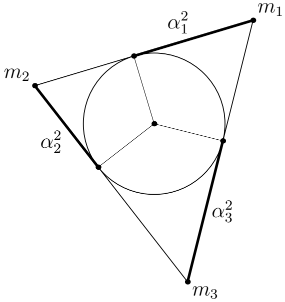

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Scaling time

Then make more time...
Define a new time variable τ such that

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = g(\boldsymbol{\alpha}) = (\alpha_2^2 + \alpha_3^2)(\alpha_3^2 + \alpha_1^2)(\alpha_1^2 + \alpha_2^2).$$

Define new Hamiltonian

$$K = (H - h)g(\boldsymbol{\alpha}),$$

where H is now in new coordinates and momenta, and h is fixed to the value of H when ICs are chosen.
Physical orbits now only for $K \equiv 0$ and K is polynomial of degree 6 in $\{\alpha_j, \pi_j\}$.
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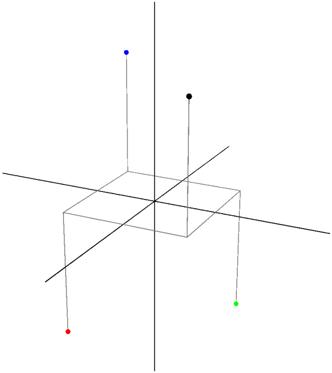
4-fold cover

New coordinates induce a 4-fold covering of shape space.

Signed area of triangle is $S = \alpha_1\alpha_2\alpha_3\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$.

Recall: $|X_l - X_k| = \alpha_k^2 + \alpha_l^2$.

So given a point $(\alpha_1, \alpha_2, \alpha_3)$, the points $(\alpha_1, -\alpha_2, -\alpha_3)$, $(-\alpha_1, \alpha_2, -\alpha_3)$ and $(-\alpha_1, -\alpha_2, \alpha_3)$ map to the same shape and orientation triangle.



New symmetry group

New Hamiltonian is

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$$K_0 = \boldsymbol{\pi}^t B\boldsymbol{\pi} - \sum m_k m_l a_k a_l, \text{ and}$$
$$B = \begin{pmatrix} B_1 & A_3 & A_2 \\ A_3 & B_2 & A_1 \\ A_2 & A_1 & B_3 \end{pmatrix}, \text{ with}$$
$$A_j = -\frac{a_j}{m_j}\alpha_k\alpha_l,$$
$$B_j = \frac{a_j}{m_j}\alpha^2 + \frac{a_k}{m_k}\alpha_l^2 + \frac{a_l}{m_l}\alpha_k^2 \text{ and}$$
$$\alpha^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2,$$

which has all the original symmetries plus four new ones that act as the identity in original coordinates.

Regularised discrete symmetry group

Old symmetries act nicely on regularised coordinates, sending $(\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3)$ to:

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\begin{aligned}
\tau &: (\alpha_1, \alpha_2, \alpha_3, -\pi_1, -\pi_2, -\pi_3) \\
\rho &: (-\alpha_1, -\alpha_2, -\alpha_3, -\pi_1, -\pi_2, -\pi_3) \\
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Full group is $C_2 \times C_2 \times S_4$, order 96.

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Change of scenery: symbolic dynamics. What is it?

- Introduce an “alphabet”.
- Assign letters to certain states of the system.
- Dynamics induce a grammar.

Reduces a continuous system to a discrete one.

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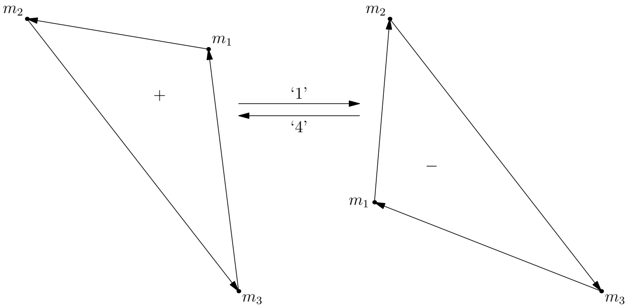
Symbolic dynamics for the 3-body problem

- Consider an orbit's projection onto the shape sphere.
- Assign symbols when passing through the equator (collinearities).
- If m_1 passes between m_2 and m_3 from upper half (positive orientation) to lower half (negative orientation) record a 1.
- Ditto from negative to positive, record a 4.
- Et cetera.



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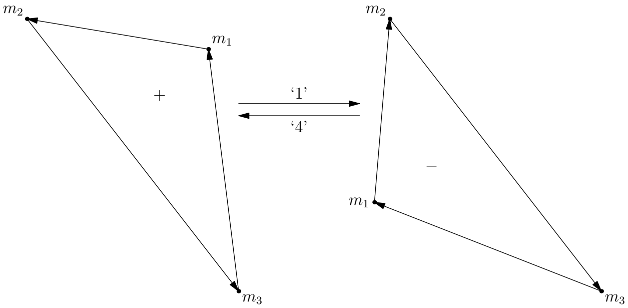

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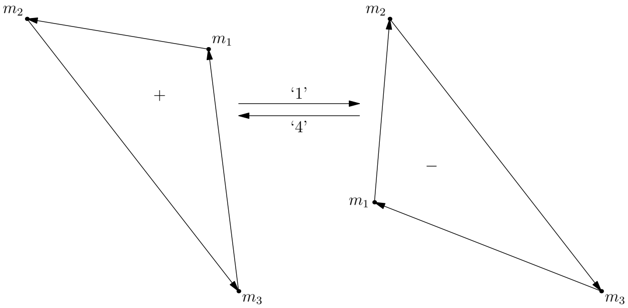

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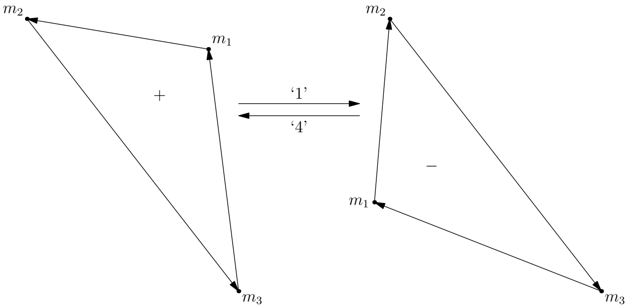

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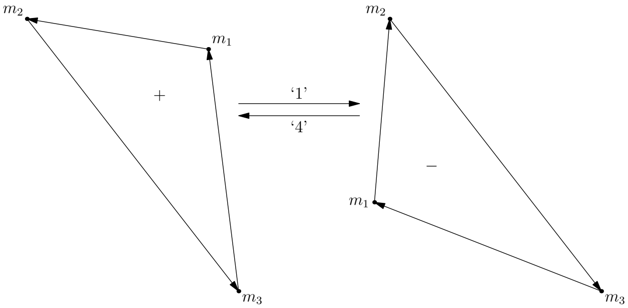
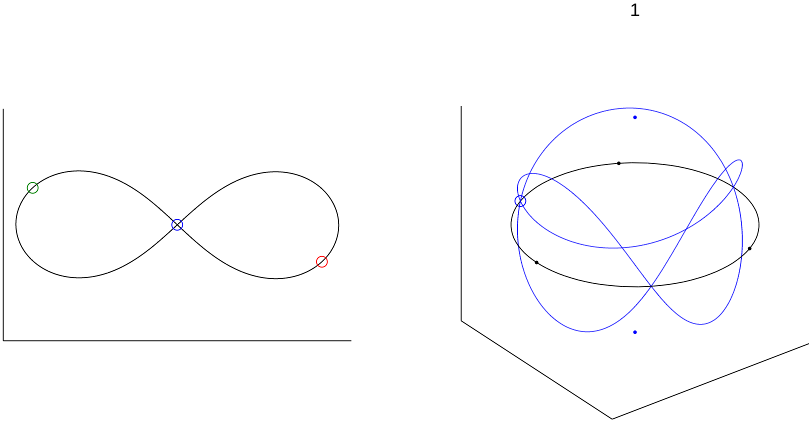
- Ditto from negative to positive, record a 4.
- Et cetera.



Example: figure-8 choreography



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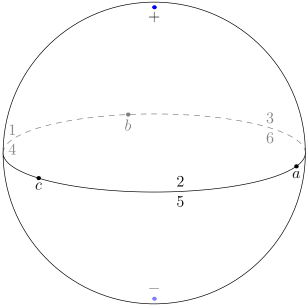

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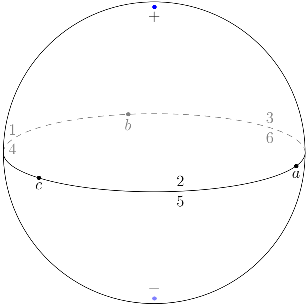

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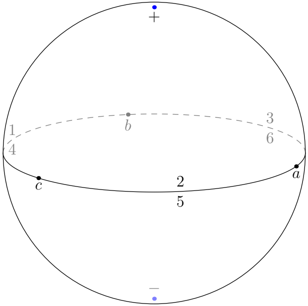

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- Each collinear configuration has two “flavours” for orientation.
- Might make sense to use six symbols: a, b, c, d, e, f.
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Symmetries of the symbolic dynamics

Certain operations result in other valid symbol sequences. Define:

- t: read a sequence from back to front. E.g. $153426 \to 624351$.
- r: reverse the orientation of each symbol. E.g. $1a4a \to 4a1a$.
- σ_j: same as earlier, $k \leftrightarrow l$ and preserve j. E.g. $1a4a \to_{\sigma_2} 3c6c$.
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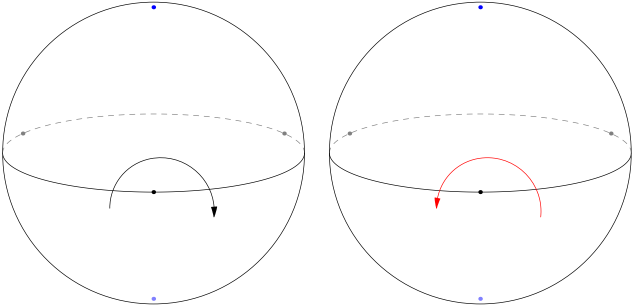
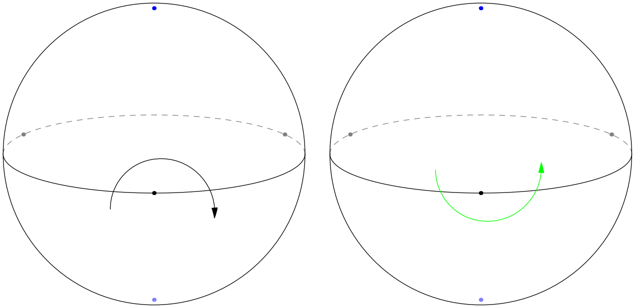
How new symmetries compare to old



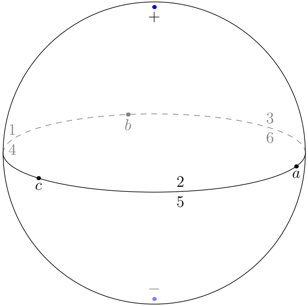
- Recall τ follows trajectory in reverse; corresponding symbol sequence is read backwards *and* orientation-reversed, corresponding to tr.
- $15 \rightarrow 24$

Symmetry comparison



- Orientation reversal ρ reflects in the equatorial plane, corresponding exactly to r.
- $15 \to 42$

Symmetry comparison



- Index-swapping σ_j reflects in the plane through collision point of k, l and equilateral points (north/south poles). Identical between frames of reference.
- Cycling c rotates trajectory by $\frac{2\pi}{3}$ ccw about axis of shape sphere. Identical between frames.

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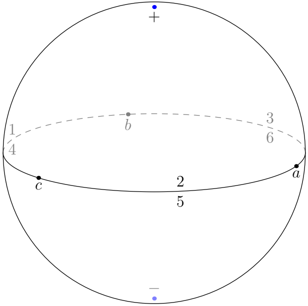

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Application: perdiodic orbits of regularised system

- By definition a periodic orbit repeats. If $z(\tau)$ is an orbit with period T at (scaled) time τ, then $z(\tau) = z(\tau + T)$.
- If $z(\tau)$ has n collinearities/collisions, symbol sequence repeats after n symbols.
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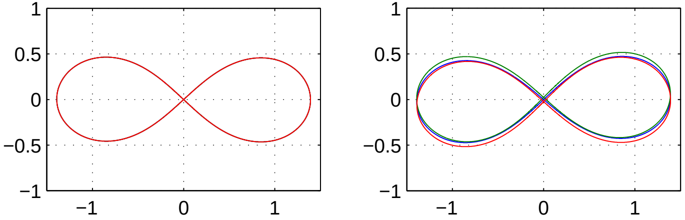
Symbol sequences are not unique!

Example: figure-8 choreography and its “shadow”.



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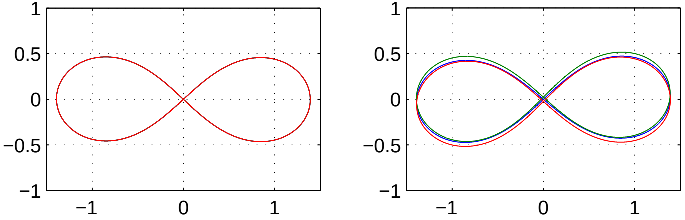

Classification by symbol sequence

- Orbits with same symbol sequence have same symmetries.
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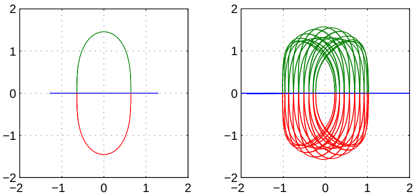

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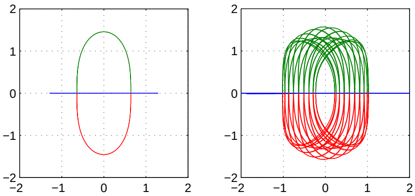

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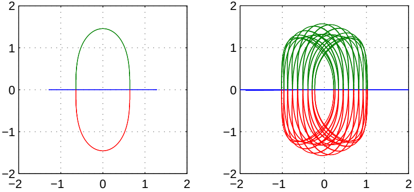

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- Orbital symmetries determined by phase space coordinates at collinear configurations.
- E.g. 153426 has 12 symmetries: $\{I, r, c, cr, c^2, c^2r, \sigma_1 t, \sigma_1 tr, \sigma_2 t, \sigma_2 tr, \sigma_3 t, \sigma_3 tr\}$.
- Figure-8 choreography has the same symmetry group: $\{I, \rho, c, c\rho, c^2, c^2\rho, \sigma_1\tau\rho, \sigma_1\tau, \sigma_2\tau\rho, \sigma_2\tau, \sigma_3\tau\rho, \sigma_3\tau\}$.
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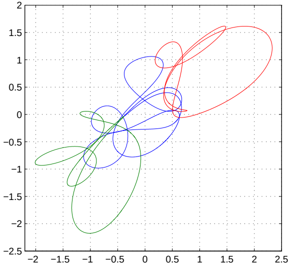
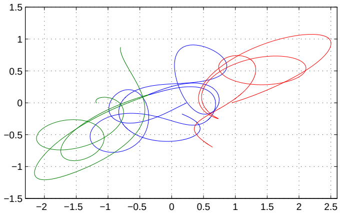
Other examples: no symmetry

Some sequences/orbits have *no* symmetries: 15151615343426



No symmetry 2

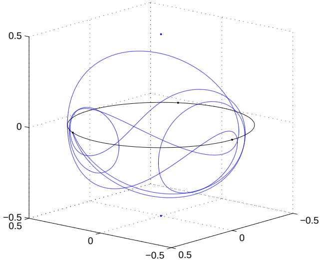
No, it actually looks like this:



Previous picture was in a rotating frame to show that it closes.

What makes this orbit really weird

- Angle between starting and finishing configurations is -0.8262.
- But "rotation angle" is -5.0152.
- Reconstruction is accurate, verified by symplectic Cartesian integration with step size control.



But there are well-behaved orbits

E.g. symbol sequence 15151624343426

- Symmetry group of symbol sequence is $\{I, t, \sigma_1 r, \sigma_1 tr\}$.
- Example from this family has symmetry group $\{I, s_1 \tau \rho\}$.

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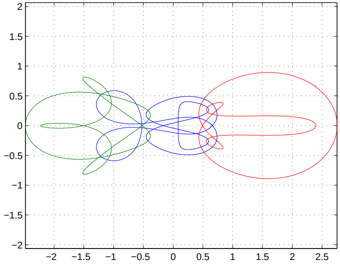

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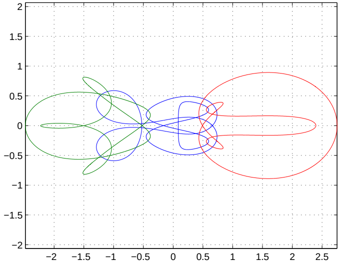

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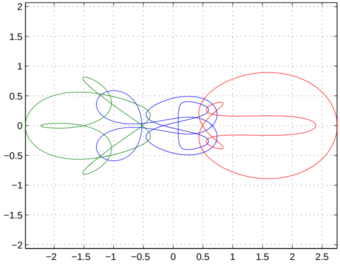
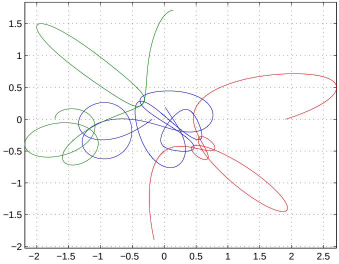
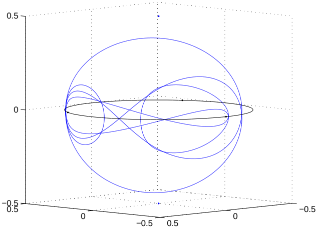


Well-bahaved orbit

This orbit has same final Cartesian angle and rotation angle: -1.654.

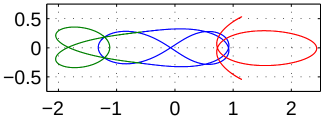


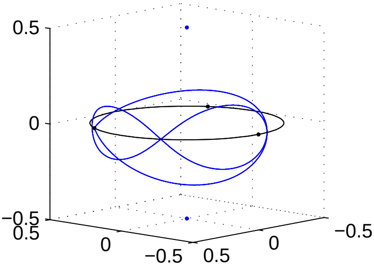


A collision orbit

Symbol sequence $1516c34243c6$.

- Symmetries $\{I, t, r, tr\}$.
- Orbit symmetry group $\{I, s_2\tau\rho, s_1\rho, s_3\tau\}$.





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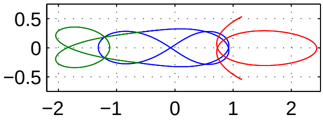

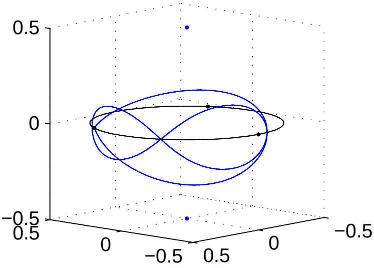

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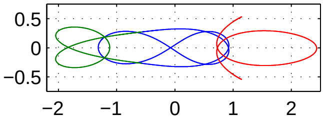

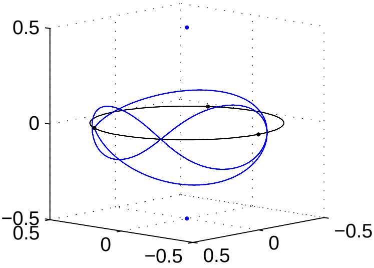

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