Binary collisions in the planar 3-body problem with vanishing angular momentum

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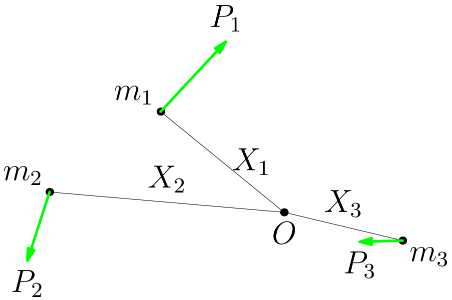
Classical planar 3-body problem

Let $X_j, P_j \in \mathbb{C}$ be the positions and momenta of three point masses $m_j \in \mathbb{R}^+$, chosen such that

$$\sum m_j X_j = \sum P_j = \sum \bar{X}_j P_j = 0$$

centre of mass, centre of momentum and vanishing angular momentum, respectively.



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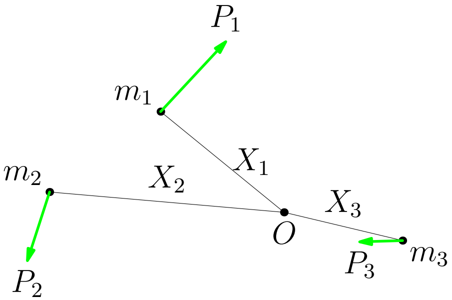

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Hamiltonian is

$$H = \sum \frac{|P_j|^2}{2m_j} - \sum \frac{m_k m_l}{|X_l - X_k|}.$$

Global dynamics are typically studied through simplified models (circular restricted 3-body problem, elliptical restricted 3-body problem, . . .), submanifolds of phase space (collinear 3-body problem, isosceles 3-body problem, equal masses, . . .), "special" orbits (periodic orbits in general, free-fall orbits, choreographies, . . .).

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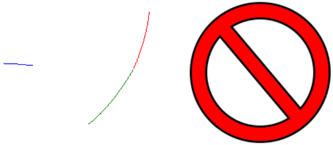
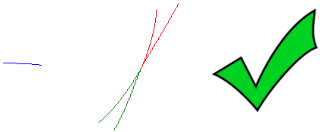
Accessing collision orbits

Normally collisions are singularities and must be removed; trajectories that result in collisions must be terminated before the collision is reached, or all ICs (may be dense?) leading to collisions must be removed.



Regularisation allows inclusion of binary collisions. Waldvogel [2] gives a simultaneous regularisation of all three binary collisions for vanishing angular momentum.



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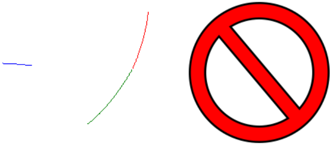

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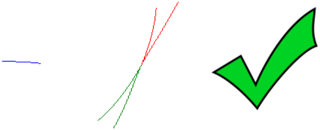

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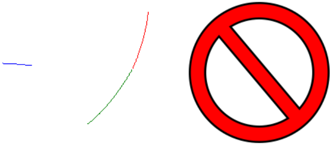

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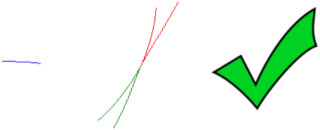
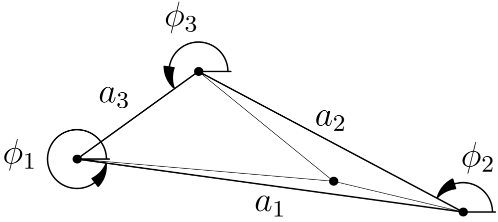


Symmetry-reduced coordinates



Reduce by rotational symmetries:

$$a_j = |X_l - X_k|$$

is the length of the side opposite m_j and

$$\phi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3)$$

a geometric rotation angle, where

$$\phi_j = \arg(X_l - X_k) \mod 2\pi$$

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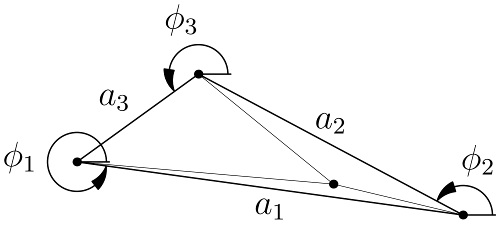

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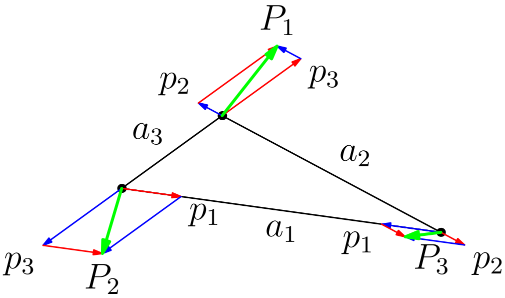
Geometry of symmetry-reduced momenta

Now obtain canonically conjugated momenta to a_j and ϕ via a generating function: $p_j, p_\phi \in \mathbb{R}$, such that

$$P_j = p_k e^{i\phi_k} - p_l e^{i\phi_l} + \frac{ip_\phi}{3}\left(\frac{e^{i\phi_k}}{a_k} - \frac{e^{i\phi_l}}{a_l}\right)$$



These are lengths of projections of the physical momenta P_j onto each side in the direction of the adjacent side when $p_\phi = 0$.

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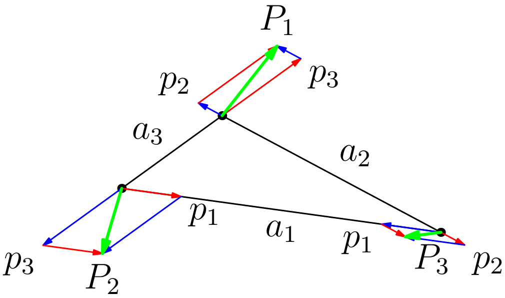

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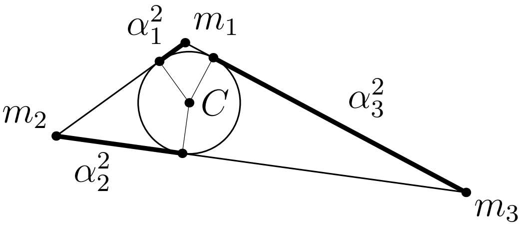
Regularisation

Now define regularised coordinates $\alpha_j \in \mathbb{R}$ such that

$$a_j = \alpha_k^2 + \alpha_l^2.$$



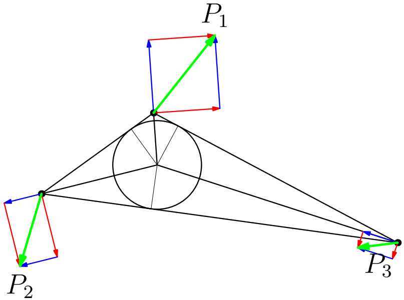
Regularised momenta

New regularised momenta are $\pi_j \in \mathbb{R}$ obtained from a generating function such that

$$p_j = \frac{1}{4}\left(-\frac{\pi_j}{\alpha_j} + \frac{\pi_k}{\alpha_k} + \frac{\pi_l}{\alpha_l}\right).$$

Relationship to physical momenta is

$$P_j = \frac{1}{4}\left(\left(e^{i\phi_k} - e^{i\phi_l}\right)\frac{\pi_j}{\alpha_j} + \left(e^{i\phi_k} + e^{i\phi_l}\right)\left(\frac{\pi_l}{\alpha_l} - \frac{\pi_k}{\alpha_k}\right)\right)$$



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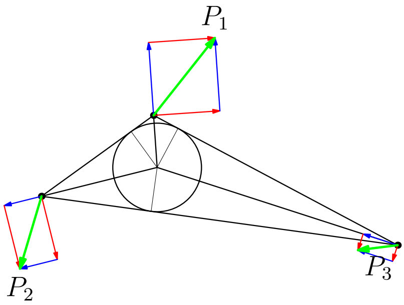

Scaled time

Introduce fictional time τ such that $\frac{dt}{d\tau} = a_1 a_2 a_3$.

Use Poincaré's trick: new Hamiltonian is

$$K = (H - h)\, a_1 a_2 a_3,$$

where h is the value of H along a solution, so $K \equiv 0$ for all physical solutions. Now a polynomial degree 6.

$$K = \pi^T B(\alpha)\,\pi - \sum m_k m_l a_k a_l - h a_1 a_2 a_3$$

when $p_\phi = 0$, where

$$B = \begin{pmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & B_1 \\ B_2 & B_1 & A_3 \end{pmatrix}$$

$$A_j = \frac{a_j}{m_j}\alpha^2 + \frac{a_k}{m_k}\alpha_l^2 + \frac{a_l}{m_l}\alpha_k^2$$

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Symmetries

Discrete symmetry group of un-regularised equal-mass system is $C_2 \times C_2 \times S_3$, order 24:

$$\sigma_j : \text{permutes indices } k \text{ and } l$$
$$c, c^2 : \text{cycle indices by 1, 2 } (c^3 = I)$$
$$\rho : \text{spatial reflection, } \rho((\alpha, \pi)) = (-\alpha, \pi)$$
$$\tau : \text{“time reflection”, } \tau((\alpha, \pi)) = (\alpha, -\pi)\,.$$

Regularisation introduces new symmetries that act as identity on physical trajectories:

s_j : swaps signs of α_k, α_l, π_k, π_l simultaneously.

e.g. $s_1((\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3)) = (\alpha_1, -\alpha_2, -\alpha_3, \pi_1, -\pi_2, -\pi_3)$

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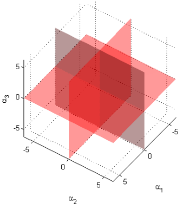
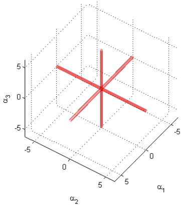
Geometry of regularised space

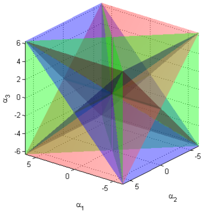
We have j-eclipse (or j-syzygy) when $\alpha_j = 0$.

Call it kl-collision when $\alpha_k = \alpha_l = 0$.





The six planes $\alpha_k^2 = \alpha_l^2$ for each pair k, l are isosceles configurations with $a_k = a_l$.



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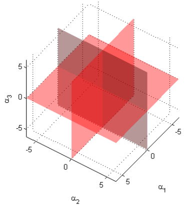

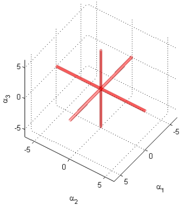

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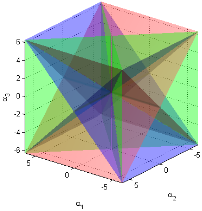

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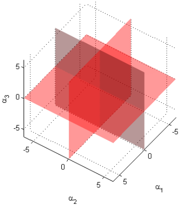

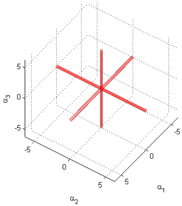

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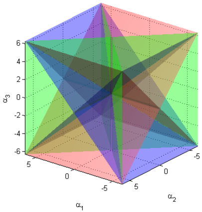

Collision constraints

Consider the kl-collision: $\alpha_k = \alpha_l = 0$ and $\alpha_j \neq 0$.

Now the regularised Hamiltonian gives constraints on π_k and π_l, but none on π_j. We have

$$\pi_k^2 + \pi_l^2 = \frac{8m_k^2 m_l^2}{m_k + m_l}$$

So let

$$\begin{aligned} \pi_k &= R\cos\theta \\ \pi_l &= R\sin\theta, \end{aligned}$$

with $R = \frac{4m_k m_l}{\sqrt{2(m_k+m_l)}}$ at collision.

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$$\pi_k^2 + \pi_l^2 = \frac{8m_k^2 m_l^2}{m_k + m_l}$$

So let

$$\begin{aligned} \pi_k &= R\cos\theta \\ \pi_l &= R\sin\theta, \end{aligned}$$

with $R = \frac{4m_k m_l}{\sqrt{2(m_k+m_l)}}$ at collision.

Vector field at 23-collision

Let $z = (\alpha_1, \alpha_2, \alpha_3, \pi_1, \pi_2, \pi_3)^T$ be the phase space of shape variables. At 23-collision, the vector field becomes

$$\dot{z} \to \begin{pmatrix} 0 \\ \frac{1}{4}\alpha_1^4 \pi_2 \left(\frac{1}{m_2} + \frac{1}{m_3}\right) \\ \frac{1}{4}\alpha_1^4 \pi_3 \left(\frac{1}{m_2} + \frac{1}{m_3}\right) \\ 0 \\ \frac{\alpha_1^3 \pi_1 \pi_2}{4m_3} \\ \frac{\alpha_1^3 \pi_1 \pi_3}{4m_2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}\alpha_1^4 \left(\frac{1}{m_2} + \frac{1}{m_3}\right) R\cos\theta \\ \frac{1}{4}\alpha_1^4 \left(\frac{1}{m_2} + \frac{1}{m_3}\right) R\sin\theta \\ 0 \\ \frac{\alpha_1^3 \pi_1 R}{4m_3} \cos\theta \\ \frac{\alpha_1^3 \pi_1 R}{4m_2} \sin\theta \end{pmatrix}$$

Recall $p_j = \frac{1}{4}\left(-\frac{\pi_j}{\alpha_j} + \frac{\pi_k}{\alpha_k} + \frac{\pi_l}{\alpha_l}\right)$ and, when p_ϕ vanishes, $P_j = p_k e^{i\phi_k} - p_l e^{i\phi_l}$.

$$\implies P_j = \frac{1}{4}\left(\left(e^{i\phi_k} - e^{i\phi_l}\right)\frac{\pi_j}{\alpha_j} + \left(e^{i\phi_k} + e^{i\phi_l}\right)\left(\frac{\pi_l}{\alpha_l} - \frac{\pi_k}{\alpha_k}\right)\right).$$

Now series expansions of $\alpha_2, \alpha_3, \pi_2, \pi_3$ with 23-collision at $\tau = 0$ give

$$\frac{\pi_2}{\alpha_2} - \frac{\pi_3}{\alpha_3} \to \frac{\pi_1 (m_2 - m_3)}{2\alpha_1 (m_2 + m_3)}$$

as $\tau \to 0$, confirming that p_2, p_3 and therefore $P_1 \to \left(e^{i\phi_2} - e^{i\phi_3}\right)\frac{\pi_1}{4\alpha_1}$ are finite. (Confirms intuition!)

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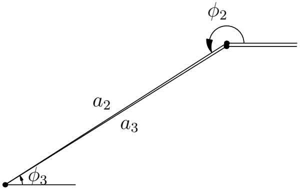

Furthermore, $\left(e^{i\phi_2} - e^{i\phi_3}\right)$ points from incentre to X_1.
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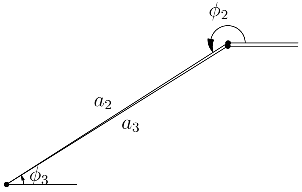

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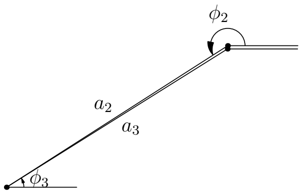

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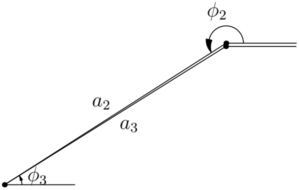

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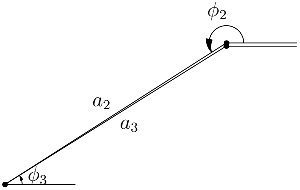

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Investigation of brake-collisions

Suppose now that at $\tau = 0$ we have $\alpha_2 = \alpha_3 = \pi_1 = 0$, $\alpha_1 \neq 0$, $\pi_2 = R\cos\theta$ and $\pi_3 = R\sin\theta$: 23-brake-collision conditions. Vector field becomes

$$\dot{z} \to \begin{pmatrix} 0 \\ \frac{1}{4}\alpha_1^4\pi_2\left(\frac{1}{m_2}+\frac{1}{m_3}\right) \\ \frac{1}{4}\alpha_1^4\pi_3\left(\frac{1}{m_2}+\frac{1}{m_3}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}\alpha_1^4\left(\frac{1}{m_2}+\frac{1}{m_3}\right)R\cos\theta \\ \frac{1}{4}\alpha_1^4\left(\frac{1}{m_2}+\frac{1}{m_3}\right)R\sin\theta \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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We have from the full vector field that

$$\begin{aligned}
\dot{\alpha}_1(\alpha_1,\alpha_2,\alpha_3,\pi_1,\pi_2,\pi_3) &= -\dot{\alpha}_1(\alpha_1,-\alpha_2,-\alpha_3,-\pi_1,\pi_2,\pi_3)\\
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and at 23-brake-collision $\alpha_2 = \alpha_3 = \pi_1 = 0$, $\alpha_1 \neq 0$, $\pi_2 = R\cos\theta$ and $\pi_3 = R\sin\theta$.

This and time reversibility of solutions to Hamilton's equations implies that α_1, π_2 and π_3 are even functions and π_1, α_2 and α_3 are odd functions about the 23-brake-collision.

I.e. acting like $s_1 \circ \tau$ symmetry, or time-reversing, as recall s_j act as the identity on physical trajectories.

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Consequences

Lemma

An orbit with two brake-collisions, two brake-points or a brake-collision and a brake-point must be periodic.

Proof.

Brake-collisions (and brake-points) cause the masses to trace backwards over their physical trajectories. Any trajectory that joins two such points can only be periodic. □

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No periodic orbit can have more than two different “types” of brake-collisions.

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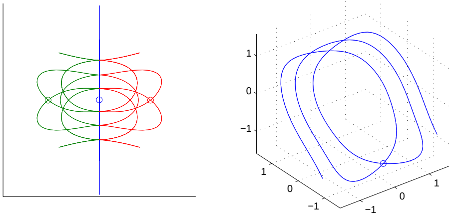
Types of periodic collision orbits: isosceles

Definition

A kl-isosceles orbit is an orbit on either of the invariant manifolds where $\alpha_k = \pm\alpha_l$ and $\pi_k = \pm\pi_l$, requiring that $m_k = m_l$.



These manifolds intersect only with the α_j-axis. I.e. $\alpha_k = \alpha_l = 0$, so only kl-collisions may (and must) occur for such orbits.

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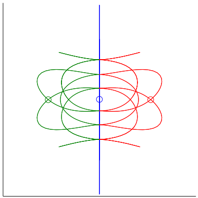

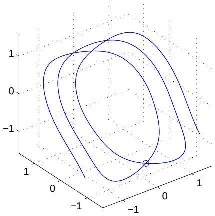

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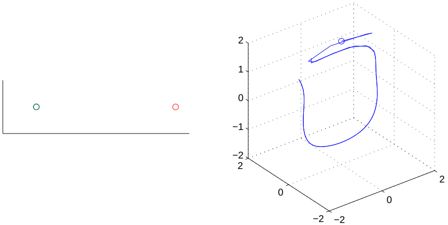
Types of periodic collision orbits: collinear

Definition

A j-collinear orbit is an orbit on the invariant manifold $\alpha_j = \pi_j = 0$.

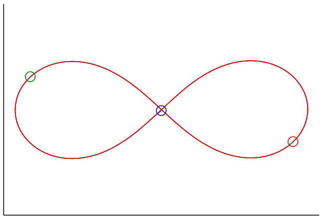


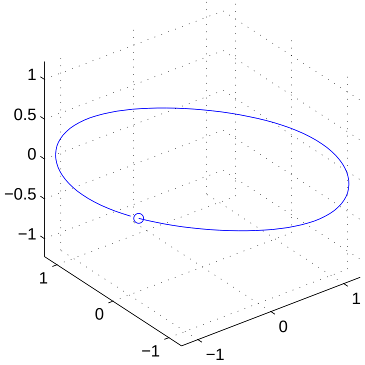
This manifold intersects with the α_k- and α_l-axes, so both jk- and jl-collisions may (must) occur.

Types of periodic collision orbits: other cases

Classify orbits by number of collisions. Possibilities are:

- Type-0: no collisions;

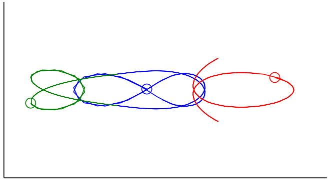


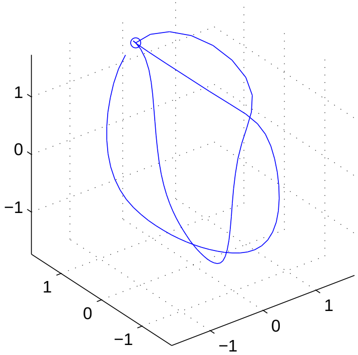


Types of periodic collision orbits: other cases

Classify orbits by number of collisions. Possibilities are:

- Type-1: trajectory passes through exactly one axis (isosceles or not);

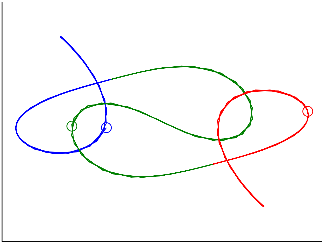


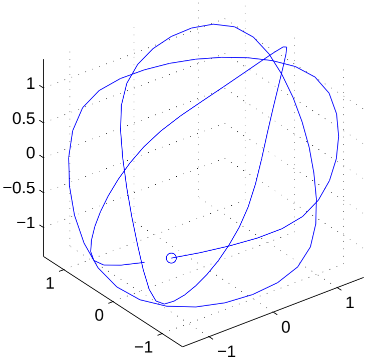


Types of periodic collision orbits: other cases

Classify orbits by number of collisions. Possibilities are:

- Type-2: trajectory passes through exactly two axes (collinear or not);





Types of periodic collision orbits: other cases

Classify orbits by number of collisions. Possibilities are:
Type-3: trajectory passes through all three axes.
So far in numerical work, only encountered types 0, 1, 2. No evidence so far that type-3 orbits exist.
Interestingly, so far no periodic collision orbit has appeared that was not a brake-collision orbit if it was not also either isosceles or collinear.

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Some numerical results

A symplectic numerical scheme exists [1] to integrate the regularised system. Using an appropriate Poincaré section and Newton's method, we find periodic orbits.

- Collinear orbits turn up commonly;
- Isosceles orbits are less common;
- Type-1 and type-2 orbits are not uncommon;
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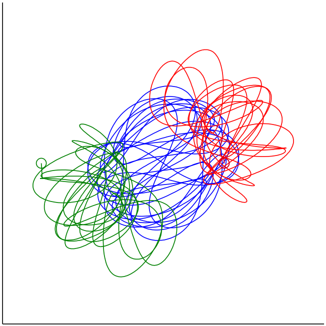

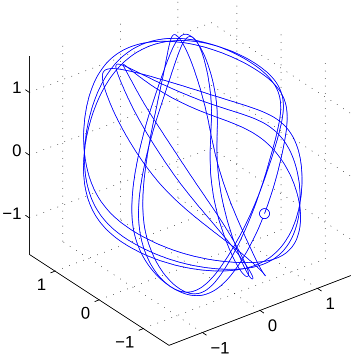

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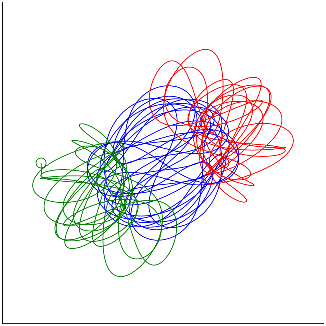

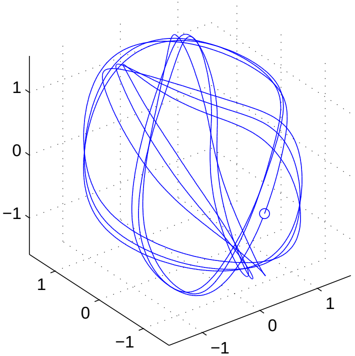

Some numerical results

A symplectic numerical scheme exists [1] to integrate the regularised system. Using an appropriate Poincaré section and Newton's method, we find periodic orbits.

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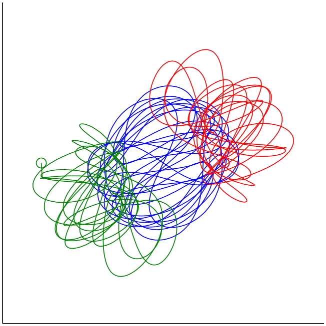

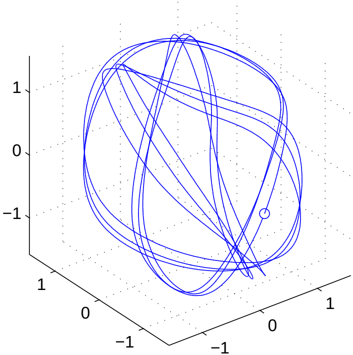

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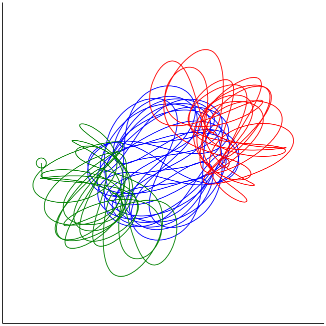

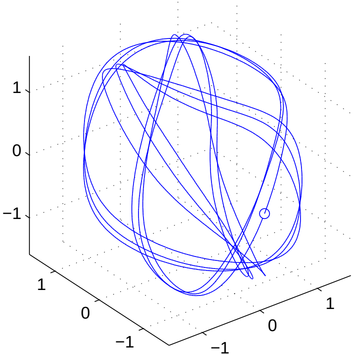

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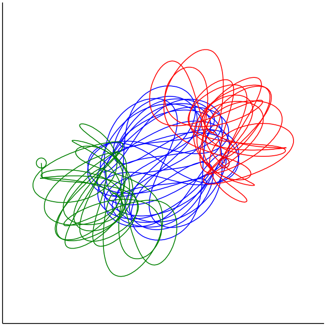

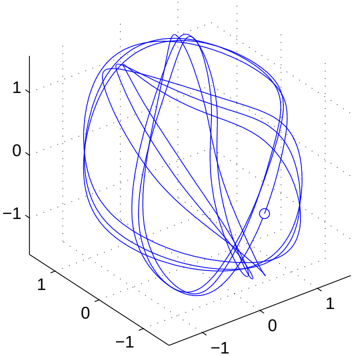

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