Inverse positivity for general Robin problems on Lipschitz domains

Daniel Daners

Abstract. It is proved that elliptic boundary value problems in divergence form can be written in many equivalent forms. This is used to prove regularity properties and maximum principles for problems with Robin boundary conditions with negative or indefinite boundary coefficient on Lipschitz domains by rewriting them as a problem with positive coefficient. It is also shown that such methods cannot be applied to domains with an outward pointing cusp. Applications to the regularity of the harmonic Steklov eigenfunctions on Lipschitz domains are given.

Mathematics Subject Classification (2000). 35J25, 35B50, 35B65.

Keywords. Elliptic boundary value problems, Robin problems, L_p-estimates, maximum priciple, Steklov eigenfunctions.

1. Introduction

Consider the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}u &= f \quad \text{in } \Omega, \\ \mathcal{B}u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

on a bounded open set $\Omega \subset \mathbb{R}^N$, where $\mathcal{A}$ is a strongly elliptic operator in divergence form with real bounded and measurable coefficients. We mostly assume that $\mathcal{B}$ is a boundary operator of Robin type

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} + b_0 u,$$

but also work with mixed or Dirichlet boundary conditions. Here $\nu_{\mathcal{A}}$ is the co-normal associated with $\mathcal{A}$ (see Section 2 for a precise definition) with b_0 possibly changing sign or negative. In most papers, b_0 is assumed to be non-negative. In

Revised version 18 Oct 2008. The final publication is available at www.springerlink.com, Archiv der Mathematik, 92(2009), 57–69, doi:10.1007/s00013-008-2918-z.

many situations this is a natural assumption, but quite often it is made because many methods fail for negative or indefinite b_0.

We show how to rewrite (1.1) in many equivalent forms. The key consequence is that every problem on a Lipschitz domain with Robin boundary conditions can be written in a form such that the new boundary coefficient b_0 is non-negative. At the same time we preserve the structure of the operators $(\mathcal{A}, \mathcal{B})$. This shows that, on a Lipschitz domain, all well known results for $b_0 \geq 0$ apply to the case with negative or indefinite boundary coefficients b_0. This includes the L_p-regularity theory from [11], maximum principles and inverse positivity as well as properties of eigenvalues and eigenfunctions. In particular we get a very simple and elementary proof and generalisation of the maximum principle in [2, Section 6]. Similar arguments can be applied to parabolic problems associated with (1.1) such as those in [4,10]. By giving a simple example we show that the construction we make is not possible by any means for domains with an outwards pointing cusp (see Example 3.4).

We further apply the theory to show that the harmonic Steklov eigenfunctions as used for instance in [5,6] are continuous up to the boundary on Lipschitz domains. We conclude the paper by some remarks on the validity of the maximum principle for Dirichlet problems for operators $\mathcal{A}$ of general structure.

2. Equivalent boundary value problems

The purpose of this section is to show how an elliptic operator and the corresponding boundary operator of Dirichlet or Neumann type can be written in equivalent forms. We assume that $\mathcal{A}$ is given by

$$\mathcal{A}u = -\operatorname{div}\bigl(A_0\nabla u + au\bigr) + b\cdot\nabla u + c_0 u \tag{2.1}$$

on some open set $\Omega \subset \mathbb{R}^N$ with $A_0 \in L_\infty(\Omega, \mathbb{R}^{N\times N})$, $a, b \in L_\infty(\Omega, \mathbb{R}^N)$ and $c_0 \in L_\infty(\Omega)$. We also assume that there exists $\alpha_0 > 0$ such that

$$\operatorname{Re}\xi^T A_0(x)\xi \geq \alpha_0|\xi|^2 \tag{2.2}$$

for all $\xi \in \mathbb{C}^N$ and almost all $x \in \Omega$. We further assume that $\partial\Omega$ is the disjoint union of the open and closed subsets Γ_0 and Γ_1 of $\partial\Omega$. To make sure there is an outward pointing unit normal ν at almost every point on Γ_1 we suppose that Γ_1 is Lipschitz. As usual we call the expression

$$\frac{\partial}{\partial\nu_{\mathcal{A}}}u := (A_0\nabla u + au)\cdot\nu$$

the *co-normal derivative* of u associated with $\mathcal{A}$. We define $\mathcal{B}$ on $\partial\Omega$ by

$$\mathcal{B}u := \begin{cases} 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial\nu_{\mathcal{A}}} + b_0 u & \text{on } \Gamma_1, \end{cases} \tag{2.3}$$

where $b_0 \in L_\infty(\Gamma_1)$. We are concerned with weak solutions of (1.1). We let

$$H^1_{\Gamma_1}(\Omega)$$

be the closure of $C_c^\infty(\Omega\cup\Gamma_1)$ in $H^1(\Omega)$. We define weak solutions as follows.

Definition 2.1. We call the map $a(\cdot\,,\cdot)\colon H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\to\mathbb{R}$ given by

$$a(u,v) := \int_\Omega (A_0\nabla u + au)\cdot\nabla v\,dx + \int_\Omega (b\cdot\nabla u + c_0u)v\,dx + \int_{\Gamma_1} b_0uv\,d\sigma$$

the *form associated with* $(\mathcal{A},\mathcal{B})$. Moreover, a function $u\in H^1_{\Gamma_1}(\Omega)$ is called a *weak solution* of (1.1) if

$$a(u,v) = \langle f,v\rangle := \int_\Omega fv\,dx \tag{2.4}$$

for all $v\in H^1_{\Gamma_1}(\Omega)$.

Every sufficiently smooth solution is a weak solution. The idea is to multiply the first equation in (1.1) by a function $v\in C^1(\Omega)$ with $v=0$ on Γ_0. Using the divergence theorem and the boundary conditions (2.4) follows.

Given a vector field $d\in W^1_\infty(\Omega,\mathbb{R}^N)$ we define operators $(\mathcal{A}_d,\mathcal{B}_d)$ by setting

$$\mathcal{A}_du := -\operatorname{div}\big(A_0\nabla u + (a+d)u\big) + (b+d)\cdot\nabla u + (c_0+\operatorname{div} d)u$$

and

$$\mathcal{B}_du := \begin{cases} 0 & \text{on } \Gamma_0,\\ \dfrac{\partial u}{\partial\nu_{\mathcal{A}_d}} + (b_0 - d\cdot\nu)u & \text{on } \Gamma_1.\end{cases}$$

Note that the operators $(\mathcal{A}_d,\mathcal{B}_d)$ have the same structure as $(\mathcal{A},\mathcal{B})$. Our key observation is the following theorem.

Theorem 2.2. *Suppose that $d\in W^1_\infty(\Omega,\mathbb{R}^N)$. Then $u\in H^1_{\Gamma_1}(\Omega)$ is a weak solution of (1.1) if and only if u is a weak solution of*

$$\begin{aligned} \mathcal{A}_du &= f \quad \text{in } \Omega,\\ \mathcal{B}_du &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

Moreover, the forms associated with $(\mathcal{A}_d,\mathcal{B}_d)$ and $(\mathcal{A},\mathcal{B})$ are the same, that is,

$$\begin{aligned} a(u,v) = a_d(u,v) &:= \int_\Omega \big(A_0\nabla u + (a+d)u\big)\cdot\nabla v\,dx\\ &\quad + \int_\Omega \big((b+d)\cdot\nabla u + (c_0+\operatorname{div} d)u\big)v\,dx + \int_{\Gamma_1}(b_0 - d\cdot\nu)uv\,d\sigma \end{aligned}$$

for all $u,v\in H^1_{\Gamma_1}(\Omega)$.

Proof. Clearly $a_d(\cdot\,,\cdot)$ is the form associated with $(\mathcal{A}_d,\mathcal{B}_d)$, so we only need to prove that $a(u,v)=a_d(u,v)$ for all $u,v\in H^1_{\Gamma_1}(\Omega)$. Rearranging terms we get

$$a_d(u,v) - a(u,v) = \int_\Omega d\cdot(u\nabla v + v\nabla u) + (\operatorname{div} d)uv\,dx - \int_{\Gamma_1}(d\cdot\nu)uv\,d\sigma,$$

so we need to show that the right hand side of the above equation is zero. By the product rule for Sobolev functions (see [14, Section 4.2.2])

$$\operatorname{div}(duv) = (\operatorname{div} d)uv + d\cdot(u\nabla v + v\nabla u).$$

If $v \in C_c^1(\Omega \cup \Gamma_1)$, then the divergence theorem for Lipschitz domains (see [14, Section 4.3]) implies that

$$\int_\Omega \operatorname{div}(duv)\,dx = \int_{\Gamma_1} (d\cdot\nu)uv\,d\sigma = \int_\Omega d\cdot(u\nabla v + v\nabla u) + (\operatorname{div} d)uv\,dx.$$

By the density of $C_c^1(\Omega\cup\Gamma_1)$ in $H^1_{\Gamma_1}(\Omega)$ and the continuity of the trace operator from $H^1_{\Gamma_1}(\Omega)$ into $L_2(\Gamma_1)$, the above identity holds for $v \in H^1_{\Gamma_1}(\Omega)$. This completes the proof of the theorem. □

Remark 2.3. Note that the arguments in the above proof work for every domain admitting the divergence theorem, not just Lipschitz domains.

We next get some properties of the form associated with $(\mathcal{A},\mathcal{B})$. They are well known, but for completeness we include the short proof.

Proposition 2.4. *If $b_0 \geq 0$, then $a(\cdot\,,\cdot)\colon H^1_{\Gamma_1}(\Omega)\times H^1_{\Gamma_1}(\Omega)\to\mathbb{R}$ is bounded, and*

$$a(u,u) + \lambda\|u\|_2^2 \geq \frac{\alpha_0}{2}\|\nabla u\|_2^2 \tag{2.6}$$

for all $u \in H^1_{\Gamma_1}(\Omega)$ and all

$$\lambda \geq \lambda_0 := \frac{\|a+b\|_\infty^2}{2\alpha_0} + \|c_0^-\|_\infty.$$

Proof. To get the boundedness we use the trace inequality (see [19, Théorème 4.2] or [14, Section 4.3]) to estimate the boundary term. For (2.6) we use the ellipticity condition (2.2) and the estimate

$$\begin{aligned}
\alpha_0\|\nabla u\|_2^2 &\leq a(u,u) - \int_\Omega (a+b)\cdot u\nabla u - c_0^- u^2\,dx \\
&\leq a(u,u) + \|a+b\|_\infty\|u\|_2\|\nabla u\|_2 + \|c_0^-\|_\infty\|u\|_2^2 \\
&\leq a(u,u) + \frac{\alpha_0}{2}\|\nabla u\|_2^2 + \Big(\frac{\|a+b\|_\infty^2}{2\alpha_0} + \|c_0^-\|_\infty\Big)\|u\|_2^2.
\end{aligned}$$

In the first inequality we used that $b_0 \geq 0$ and the last follows from the elementary inequality $\xi\eta \leq (\xi^2+\eta^2)/2$ for all $\xi,\eta\geq 0$. □

In the above proof it is essential to assume that $b_0 \geq 0$ if $\Gamma_1 \neq \emptyset$. However, if there exists a vector field d as in Theorem 2.2 such that $b_0 - d\cdot\nu \geq 0$, then the above result still applies.

Corollary 2.5. *Suppose that $b_0 \in L_\infty(\Gamma_1)$ and that there exists $d \in W^1_\infty(\Omega,\mathbb{R}^N)$ such that $b_0 - d\cdot\nu \geq 0$ almost everywhere on Γ_1. Then (2.6) holds for all*

$$\lambda \geq \lambda_d := \frac{\|a+b+2d\|_\infty^2}{2\alpha_0} + \|(c_0 + \operatorname{div} d)^-\|_\infty. \tag{2.7}$$

Proof. By Theorem 2.2 we have $a(u,u) = a_d(u,u)$ and therefore the assertion of the corollary follows by applying Proposition 2.4 to the form $a_d(u,u)$. □

In Section 3 we show the existence of a vector field d as required above. Before doing so we state some consequences. Let $A \in \mathcal{L}\big(H_{\Gamma_1}(\Omega), (H_{\Gamma_1}(\Omega))'\big)$ be the operator induced by the form $a(\cdot\,,\cdot)$, that is, A is given by

$$\langle Au, v\rangle := a(u, v)$$

for all $u, v \in H_{\Gamma_1}(\Omega)$. We get the following facts on the resolvent set of A considered as a closed operator on $L_2(\Omega)$.

Theorem 2.6. *Suppose that $b_0 \in L_\infty(\Gamma_1)$ is arbitrary and that there exists $d \in W^1_\infty(\Omega, \mathbb{R}^N)$ such that $b_0 - d \cdot \nu \geq 0$ almost everywhere on Γ_1. Then $(\lambda_d, \infty) \subset \varrho(-A)$, where λ_d is defined by (2.7). Moreover, A has compact resolvent as an operator on $L_2(\Omega)$.*

Proof. The first assertion follows from Corollary 2.5 together with the Lax-Milgram Theorem (see [13, Section VI.3.2.5]). Since Γ_1 is Lipschitz, Rellich's Theorem guarantees that the embedding $H_{\Gamma_1}(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Hence the resolvent $(\lambda I + A)^{-1}$ is compact as an operator on $L_2(\Omega)$. □

Remark 2.7. Depending on the vector field d we choose in the above theorem we get different upper bounds for the principal eigenvalue λ_1 of $-A$. More precisely

$$\lambda_1 \leq \inf \lambda_d,$$

where the infimum is taken over all $d \in W^1_\infty(\Omega, \mathbb{R}^N)$ with $b_0 - d \cdot \nu \geq 0$ almost everywhere on Γ_1.

3. Inverse positivity and maximum principles for Robin problems

In this section we extend the maximum principle and inverse positivity property of the resolvent of Robin problems with no conditions on the sign of b_0 from Amann [2, Section 6] to a larger class of domains and operators. The idea is to construct an equivalent problem, where the new b_0 is positive. In [2] and also by an improved method in [3, Appendix B] this is achieved by an extension of boundary values. We use a different idea. Given operators $(\mathcal{A}, \mathcal{B})$, we construct a vector field d supported near Γ_1 such that $b_0 - d \cdot \nu > 0$, and then apply the usual weak maximum principle. Throughout this section we use the setup and the assumptions from Section 2. We consider solutions of the differential inequality

$$\begin{aligned} \mathcal{A}u + \lambda u &\geq 0 \quad \text{in } \Omega,\\ \mathcal{B}u &\geq 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

These inequalities are to be understood in the weak sense, namely that

$$a(u, v) + \lambda\langle u, v\rangle \geq 0$$

for all $v \in H^1_{\Gamma_1}(\Omega)$ non-negative. We now state our main theorem of this section.

Theorem 3.1. *Suppose that Ω satisfies the assumptions stated in the previous section and that $b_0 \in L_\infty(\Gamma_1)$. Then there exists $\lambda^* \geq 0$ such that if $\lambda > \lambda^*$ and $u \in H^1_{\Gamma_1}(\Omega)$ satisfy (3.1), then $u \geq 0$. Moreover, if Ω is connected and $u \neq 0$, then for every compact subset K of Ω there exists a constant $c_K > 0$ such that $u(x) > c_K$ for almost all $x \in K$. In particular $u(x) > 0$ for almost all $x \in \Omega$.*

Proof. Let $\Gamma_1 \neq \emptyset$. We show in Lemma 3.2 below that there exists a vector field $d \in C^\infty(\bar{\Omega}, \mathbb{R}^N)$ such that $d \cdot \nu \geq 1$ on Γ_1. Hence there exists a constant $\beta \geq 0$ such that $b_0 + \beta d \cdot \nu \geq 0$. We then use Corollary 2.5 asserting that (2.6) holds for all

$$\lambda \geq \lambda^* := \frac{\|a + b - 2\beta d\|_\infty^2}{2\alpha_0} + \|(c_0 - \beta \operatorname{div} d)^-\|_\infty.$$

Also note that $a(u^+, u^-) = a(u^-, u^+) = 0$. Taking u^- as a test function we conclude from (2.6) and the assumptions that for $\lambda > \lambda^*$

$$(\lambda - \lambda^*)\|u^-\|_2^2 \leq a(u^-, u^-) + \lambda\|u^-\|_2^2 = -a(u, u^-) - \lambda\langle u, u^-\rangle \leq 0.$$

Hence $u^- = 0$, proving that $u \geq 0$. The latter argument also applies to the Dirichlet problem since then (2.6) also holds in that case. Given that $u \geq 0$ we can apply the weak Harnack inequality. Let $B_r(x)$ denote a ball of radius r centred at x such that $B_{4r}(x) \subset \Omega$. By [15, Theorem 8.18], there exists a constant $c_r > 0$ such that

$$\|u\|_{L_1(B_{2r}(x))} \leq c_r \operatorname*{ess-inf}_{y \in B_r(x)} u(y). \tag{3.2}$$

If $u \neq 0$, then there exists a set $U \subset \Omega$ of positive measure such that $u > 0$ on U. Hence we can choose a ball $B_{2r}(x)$ such that $\|u\|_{L_1(B_{2r}(x))} > 0$, so the above weak Harnack inequality implies that $u > 0$ almost everywhere on $B_r(x)$. Hence the set

$$U := \{x \in \Omega \colon \text{there exists } r > 0 \text{ with } B_{4r}(x) \subset \Omega \text{ and } u > 0 \text{ on } B_r(x)\}$$

is non-empty. As a union of open balls U is open. Furthermore, if we assume that $U \neq \Omega$, then by the connectedness of Ω there must exist some $x \in \partial U \cap \Omega$. Hence there exists $r > 0$ such that $B_{4r}(x) \subset \Omega$. Because $u > 0$ almost everywhere on $B_r(x) \cap U$ we have $\|u\|_{L_1((B_{2r}(x))} > 0$ and so by the weak Harnack inequality $u > 0$ on $B_r(x)$. This shows that $x \in U \cap \partial U$ which is impossible since U is open. Hence $u(x) > 0$ for almost all $x \in \Omega$. If $K \subset \Omega$ is compact, then there exist $x_i \in K$ and $r_i > 0$ $(i = 1, \dots, n)$ such that $\bigcup_{i=1}^n B_{r_i}(x_i) \supset K$. Hence by (3.2) we get

$$\operatorname*{ess-inf}_{x \in K} u(x) \geq \min_{i=1,\dots,n} \frac{1}{c_{r_i}} \|u\|_{L_1(B_{2r_i}(x_i))} =: c_K > 0,$$

completing the proof of the theorem. □

The above theorem implies that parabolic problems with negative or indefinite b_0 admit a heat kernel which is strictly positive by applying the results in [10] for instance. We finally establish the key lemma needed to prove Theorem 3.1.

Lemma 3.2. *There exists a vector field $d \in C^\infty(\bar{\Omega}, \mathbb{R}^N)$ such that $d \cdot \nu > 1$ almost everywhere on Γ_1. Given any neighbourhood of Γ_1, that vector field can be chosen such that its support lies in that neighbourhood.*

Proof. For $\delta > 0$ set $R_\delta := \{x' \in \mathbb{R}^{N-1} : |x'| < \delta\}$. By assumption the boundary Γ_1 is Lipschitz. This means that for every point $x_0 \in \Gamma_1$ there exist an orthogonal transformation T, constants $\varepsilon, \delta > 0$ and a Lipschitz function $\varphi \colon R_\delta \to \mathbb{R}$ such that

$$U := T^{-1}\bigl(\bigl\{(x', x_N) \in R_\delta \times \mathbb{R} : |x_N - \varphi(x')| < \varepsilon\bigr\}\bigr)$$

is a neighbourhood of x_0 and

$$\Omega \cap U = T^{-1}\bigl(\bigl\{(x', x_N) \in R_\delta \times \mathbb{R} : 0 < x_N - \varphi(x') < \varepsilon\bigr\}\bigr).$$

By Rademacher's theorem (see [14, Section 3.1.2]) and since φ is Lipschitz continuous, the normal to $T(\Gamma_1)$ is given by

$$\nu(x', \varphi(x')) = \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + |\nabla\varphi(x')|^2}}$$

for almost all $x' \in R_\delta$. If we define $d_0 := (0, \dots, 0, -1)$, then

$$d_0 \cdot \nu(x', \varphi(x')) = \frac{1}{\sqrt{1 + |\nabla\varphi(x')|^2}}.$$

If L is the Lipschitz constant for φ and $x' \in R_\delta$, then for every unit vector $y' \in \mathbb{R}^{N-1}$

$$\frac{|\varphi(x') - \varphi(x' + ty')|}{|t|} \leq L$$

for all t sufficiently small. Hence, $|\partial\varphi/\partial x_i(x')| \leq L$ for $i = 1, \dots, N-1$ if $t \to 0$ at every point x' where φ is differentiable. Since this is the case almost everywhere

$$d_0 \cdot \nu(x', \varphi(x')) = \frac{1}{\sqrt{1 + |\nabla\varphi(x')|^2}} \geq \frac{1}{\sqrt{1 + NL^2}}$$

for almost all $x \in R_\delta$. Since Γ_1 is compact we can cover it by finitely many open sets such as the above. Let T_i and φ_i be the corresponding maps and $U_i = T^{-1}\bigl(\bigl\{(x', x_N) \in R_\delta \times \mathbb{R} : |x_N - \varphi(x')| < \varepsilon\bigr\}\bigr)$, $i = 1, \dots, n$, the corresponding open sets as constructed above. Now let ψ_i, $i = 1, \dots, n$, be a smooth partition of unity subordinate to the covering (U_i) of Γ_1. Define the vectors $d_i := T_i^{-1} d_0$. Then

$$d(x) := \sum_{i=1}^{n} d_i \psi_i(x)$$

is in $C^\infty(\mathbb{R}^N)$. By the above and the orthogonality of T_i

$$d(x) \cdot \nu(x) = \sum_{i=1}^{n} d_i \cdot \nu(x) \psi_i(x) \geq \sum_{i=1}^{n} \frac{\psi_i(x)}{\sqrt{1 + NL^2}} = \frac{1}{\sqrt{1 + NL^2}}$$

for almost all $x \in \Gamma_1$ if we let L be the maximal Lipschitz constant of φ_i, $i = 1, \dots, n$. We can use a cutoff function for Γ_1 to make sure that d has support in a given neighbourhood of Γ_1. To complete the proof we replace d by $d\sqrt{1 + NL^2}$. $\square$

Remark 3.3. Different approaches could be taken to obtain (2.6) for λ large enough. Firstly, the compactness of the trace operator can be used as in [1] or [5, Corollary 3.5]. Alternatively, u can be replaced by $e^{\psi}u$ for a suitable function ψ as demonstrated in in [17, Section 2.1], at least if Ω is piecewise C^1. A similar method is used in [18, Proposition 3.4] for domains of class $C^{2,\alpha}$. Our method seems particularly simple and works for Lipschitz domains and general operators in divergence form with bounded and measurable coefficients.

We finally show that it is impossible to construct a vector field d with the properties in Lemma 3.2 for very simple domains which are not Lipschitz.

Example 3.4. We give an example that (2.6) and therefore Lemma 3.2 fails for a bounded domain with an exponential outward pointing cusp with endpoint x_0. Note that such a domain admits the divergence (Gauss-Green) theorem. To see this note that $\{x_0\}$ has capacity zero, so by the smoothness of the rest of the domain $\{u|_\Omega\colon u \in C_c^\infty(\mathbb{R}^N)\}$ is dense in $H^1(\Omega)$. Since Ω has finite perimeter the Gauss-Green theorem applies for smooth functions (see [14, Section 5.8]). By density of the smooth functions in $H^1(\Omega)$ the Gauss-Green formula holds for functions in $H^1(\Omega)$ as well. Hence Theorem 2.2 applies, but as we will show not Lemma 3.2. Also, every $u \in H^1(\Omega)$ has a unique trace in $L_{2,\mathrm{loc}}(\partial\Omega \setminus \{x_0\})$, but there is no trace inequality. More precisely, there is no constant $c > 0$ such that

$$\|u\|_{L_2(\partial\Omega)} \leq c\|u\|_{H^1(\Omega)} \tag{3.3}$$

for all $u \in H^1(\Omega)$. As shown in [11, Remark 3.5(f)], if $b_0 > \beta$ for some constant $\beta > 0$, then we can work with the space

$$V := \{u \in H^1(\Omega)\colon u|_{\partial\Omega} \in L_2(\Omega)\}$$

with the norm $\|u\|_V := (\|\nabla u\|_2^2 + \|u\|_{L_2(\partial\Omega)}^2)^{1/2}$. The space V is a proper subspace of $H^1(\Omega)$ and the norm $\|\cdot\|_V$ is stronger than the usual H^1-norm. On the other hand, V is a dense subspace of $H^1(\Omega)$ because $\{x_0\}$ is a set of capacity zero.

For the Robin problem we could try to work with V also in case of $b_0 < 0$. However, we show that then (2.6) cannot hold for a domain as the above. For simplicity consider $\mathcal{A} := -\Delta$. Let $b_0 \in L_\infty(\partial\Omega)$ with $b_0 \leq -\beta < 0$ for some constant $\beta > 0$. If we assume that (2.6) holds for some $\lambda > 0$, then

$$0 \leq \|\nabla u\|_2^2 + \lambda\|u\|_2^2 - \beta\|u\|_{L_2(\partial\Omega)}^2$$

for all $u \in V$. Rearranging we conclude that

$$\|u\|_{L_2(\partial\Omega)} \leq \beta^{-1}\big(\|\nabla u\|_2^2 + \lambda\|u\|_2^2\big)$$

for all $u \in V$. As V is dense in $H^1(\Omega)$ the above inequality holds for all $u \in H^1(\Omega)$. Hence a trace inequality of the form (3.3) is valid for all $u \in H^1(\Omega)$. Since this is a contradiction, (2.6) and the assertion of Lemma 3.2 cannot be true for very simple non-smooth domains admitting the divergence theorem.

4. Regularity properties for Robin and Steklov problems

Again suppose Ω and the operators $(\mathcal{A},\mathcal{B})$ satisfy the assumptions from Section 2. We consider the global regularity of solutions to the boundary value problem

$$\begin{aligned} \mathcal{A}u+\lambda u &= f \quad \text{in } \Omega,\\ \mathcal{B}u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

with $\Gamma_1 \neq \emptyset$. As we saw in Theorem 2.2 and Lemma 3.2, we can assume without loss of generality that there exists a constant $\beta > 0$ such that $b_0 \geq \beta$ on Γ_1. Since Γ_1 is Lipschitz continuous we get that

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L_{2N/(N-2)}(\Omega)$$

if $N \geq 3$ and $H^1_{\Gamma_1}(\Omega) \hookrightarrow L_q(\Omega)$ for all $q \in (1,\infty)$ if $N = 2$. Considering the resolvent $R(\lambda,-A)$ for the above problem the first assertion of the following theorem follows from [12] or [11]. We also generalise a result from [21, Section 3]. The improvement is that we do not require b_0 to be positive and bounded away from zero, and allow $p > N/2$ rather than $p > N$. We do this by using results from this paper to show that the solution of (4.1) is bounded and then apply results from [21].

Theorem 4.1. *Suppose Ω satisfies the assumptions stated in the previous section with $\Gamma_1 \neq \emptyset$. Then $R(\lambda,-A) \in \mathcal{L}\big(L_p(\Omega), L_{m(p)}(\Omega)\big)$ for all $\lambda \in \varrho(-A)$, where*

$$m(p) = \begin{cases} \dfrac{Np}{N-2p} & \text{if } 1 < p < N/2,\\ \infty & \text{if } p > N/2. \end{cases}$$

If $\Gamma_1 = \partial\Omega$, $\mathcal{A} = -\Delta$ and $p > N/2$, then $R(\lambda,-A) \in \mathcal{L}\big(L_p(\Omega), C(\bar{\Omega})\big)$ for all $\lambda \in \varrho(-A)$.

Proof. The first assertion is proved already. Hence assume that $\mathcal{A} = \Delta$ and that $f \in L_p(\Omega)$ with $p > N/2$. Then by the first assertion $u \in L_\infty(\Omega)$ and also the trace $u|_{\partial\Omega} \in L_\infty(\partial\Omega)$. Hence u solves the Neumann problem $-\Delta u = -\lambda u + f$ in Ω and $\partial u/\partial\nu = -b_0 u$ on $\partial\Omega$. Moreover, since u is a weak solution we get

$$\int_\Omega \nabla u \nabla\varphi \, dx = \int_\Omega (-\lambda u + f)\varphi \, dx - \int_{\partial\Omega} b_0 u \varphi \, d\sigma$$

for all $\varphi \in C^1(\bar{\Omega})$. Setting $\varphi = 1$ we get

$$0 = \int_\Omega (-\lambda u + f) \, dx + \int_{\partial\Omega} (-b_0 u) \, d\sigma$$

Setting $g := -b_0 u \in L_\infty(\partial\Omega)$ we can now apply [21, Corollary 2.8], which is based on [8, Theorem 5.3]), to conclude that $u \in C(\bar{\Omega})$. $\square$

From the above we get regularity of eigenfunctions. Note that we know already from that Theorem 2.6 that the above problem has compact resolvent. Hence the spectrum of $-A$ consists of eigenvalues of finite algebraic multiplicity.

Corollary 4.2. *All eigenfunctions of*

$$\begin{aligned} \mathcal{A}u &= \lambda u && \text{in } \Omega, \\ \mathcal{B}u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

are in $L_\infty(\Omega) \cap C^\mu(\Omega)$ for some $\mu \in (0,1)$. If $\mathcal{A} = -\Delta$, then the eigenfunctions are in $C(\bar{\Omega}) \cap C^\infty(\Omega)$.

Proof. By [11, Corollary 5.5] the eigenfunctions are in $L^\infty(\Omega)$. Then the local estimates from [15, Theorem 8.24] imply that the eigenfunctions are locally Hölder continuous. The remaining assertion now follows from Theorem 4.1. □

We finally consider the regularity of the Steklov eigenfunctions on Lipschitz domains. We call $u \neq 0$ a Steklov eigenfunction if there exists $\gamma \in \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \gamma u && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

On a Lipschitz domain such a problem has a sequence of eigenvalues $0 < \gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{n-1} \leq \gamma_n \to \infty$ (see [5, 7]). The corresponding eigenfunctions are solutions of $-\Delta u = 0$ with Robin boundary conditions

$$\frac{\partial u}{\partial \nu} - \gamma u = 0.$$

Hence Corollary 4.2, together with the fact that harmonic functions are in $C^\infty(\Omega)$, implies the following regularity properties for the Steklov eigenfunctions.

Corollary 4.3. *Let Ω be a bounded Lipschitz domain. Then all Steklov eigenfunctions of* (4.2) *lie in $C^\infty(\Omega) \cap C(\bar{\Omega})$.*

Remark 4.4. Similar arguments apply to problems with Wentzell boundary conditions, that is, eigenvalue problems of the form

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

As shown in [16] that problem can be rewritten as eigenvalue problem with Robin boundary conditions in the form

$$\frac{\partial u}{\partial \nu} + \frac{\gamma - \lambda}{\beta} u = 0$$

on $\partial\Omega$. Corollary 4.2 shows that every eigenfunction of (4.3) lies in $C^\infty(\Omega) \cap C(\bar{\Omega})$.

5. A note on the maximum principle

Consider an operator of the form

$$\mathcal{A}u = -\operatorname{div}(A_0\nabla u) + b\cdot\nabla u + c_0u$$

on a bounded (connected) domain with the same assumptions as in Section 2. According to the weak maximum principle $\mathcal{A}u \geq 0$ and $u \in H_0^1(\Omega)$ imply that $u \geq 0$ in Ω if $c_0 \geq 0$ (see [15, Theorem 8.1]). Applying the same arguments as in the proof of Theorem 3.1 we conclude that

$$R(\lambda, -A) := (\lambda I + A)^{-1}$$

is a compact, positive and irreducible operator for all $\lambda \geq 0$ if A is the operator associated with $\mathcal{A}$ and Dirichlet boundary conditions on $L_2(\Omega)$. Then its dual

$$R(\lambda, -A') := \big((\lambda I + A)^{-1}\big)',$$

is compact, positive and irreducible. The dual operator on $L_2(\Omega)$ is associated with

$$\mathcal{A}^\sharp u := -\operatorname{div}(A_0^t\nabla u + bu) + c_0u$$

with $c_0 \geq 0$. Hence also that operator satisfies a weak maximum principle. Note that $R(\lambda, -A) \geq 0$ implies that $0 \geq \lambda_1$, where λ_1 is the principal eigenvalue of $\mathcal{A}$ and hence of $\mathcal{A}^\sharp$. This proves the following fact also observed in [9].

Proposition 5.1. *Let $\mathcal{A}$ be as in (2.1) with $c_0 \geq 0$ and either $a \equiv 0$ or $b \equiv 0$. Then $[0,\infty) \subset \varrho(-A)$ and $R(\lambda, -A)$ is compact, positive and irreducible for all $\lambda \geq 0$.*

We have seen that operators of the form

$$\mathcal{A}u = -\operatorname{div}(A_0\nabla u) + b\cdot\nabla u \quad \text{and} \quad \mathcal{A}u := -\operatorname{div}(A_0\nabla u + au)$$

satisfy a maximum principle. We could ask whether or not this is also the case for

$$\mathcal{A}u := -\operatorname{div}(A_0\nabla u + au) + b\cdot\nabla u \tag{5.1}$$

involving both types of first order terms. The answer is negative. We give an example that an operator of the form (5.1) with Dirichlet boundary conditions can have an arbitrary first eigenvalue even if $c_0 \geq 0$ is very large.

Example 5.2. Let $d \in C^\infty(\mathbb{R}^N)$ be a smooth vector field with $\operatorname{div} d = 1$, for instance $d(x) = (x_1, 0, \dots, 0)$ where $x = (x_1, \dots, x_N)$. By Theorem 2.2

$$\begin{aligned} -\Delta u + \lambda u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

and

$$\begin{aligned} -\operatorname{div}(\nabla u - \lambda du) - \lambda d\cdot\nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

are equivalent problems. If λ_1 is the first eigenvalue of the Dirichlet Laplacian with eigenfunction u, then

$$-\operatorname{div}(\nabla u - \mu du) - \mu d\cdot\nabla u = (\lambda_1 + \mu)u.$$

Hence for every $\mu \in \mathbb{R}$ we find an operator of the form (5.1) with $c_0 = 0$ such that $\lambda_1 + \mu$ is its first eigenvalue. Note however, the maximum principle as stated in Theorem 3.1 only applies for λ larger than the first eigenvalue. Hence given an operator of the general form (2.1), it is extremely difficult to say something about its spectral bound and the validity of a maximum principle. In fact, the example shows that it is really necessary to put additional conditions on the coefficients a, b in $\mathcal{A}$ such as those in [15, Theorem 8.1] or [20] to obtain a maximum principle.

References

[1] G. A. Afrouzi and K. J. Brown, *On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions*, Proc. Amer. Math. Soc. **127** (1999), no. 1, 125–130. MR1469392 (99b:35035)

[2] H. Amann, *Dual semigroups and second order linear elliptic boundary value problems*, Israel J. Math. **45** (1983), no. 2-3, 225–254. MR719122 (85i:35043)

[3] ______, *Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems*, Differential Integral Equations **3** (1990), no. 1, 13–75. MR1014726 (90i:35124)

[4] W. Arendt and A. F. M. ter Elst, *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Operator Theory **38** (1997), no. 1, 87–130. MR1462017 (98k:47094)

[5] G. Auchmuty, *Steklov eigenproblems and the representation of solutions of elliptic boundary value problems*, Numer. Funct. Anal. Optim. **25** (2004), no. 3-4, 321–348. MR2072072 (2005d:35187)

[6] ______, *Spectral characterization of the trace spaces $H^s(\partial\Omega)$*, SIAM J. Math. Anal. **38** (2006), no. 3, 894–905 (electronic). MR2262947 (2008:46054)

[7] C. Bandle, *Isoperimetric inequalities and applications*, Monographs and Studies in Mathematics, vol. 7, Pitman, Boston, Mass., 1980. MR0572958 (81e:35095)

[8] R. F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains*, Ann. Probab. **19** (1991), no. 2, 486–508. MR1106272 (92i:60142)

[9] F. Belgacem, *Elliptic boundary value problems with indefinite weights: variational formulations of the principal eigenvalue and applications*, Pitman Research Notes in Mathematics Series, vol. 368, Longman, Harlow, Essex, 1997. MR1478921 (98k:35026)

[10] D. Daners, *Heat kernel estimates for operators with boundary conditions*, Math. Nachr. **217** (2000), 13–41. MR1780769 (2002f:35109)

[11] ______, *Robin boundary value problems on arbitrary domains*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4207–4236. MR1650081 (2000m:35048)

[12] ______, *A priori estimates for solutions to elliptic equations on non-smooth domains*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), no. 4, 793–813.

[13] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 2*, Springer-Verlag, Berlin, 1988. MR0969367 (89m:00001)

[14] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR1158660 (93f:28001)

[15] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)

[16] J. Kennedy, *A Faber-Krahn inequality for the Laplacian with generalised Wentzell boundary conditions*, J. Evol. Equ. **8** (2008), no. 3, 557–582.

[17] A. A. Lacey, J. R. Ockendon, and J. Sabina, *Multidimensional reaction diffusion equations with nonlinear boundary conditions*, SIAM J. Appl. Math. **58** (1998), no. 5, 1622–1647 (electronic). MR1637882 (99g:35066)

[18] J. López-Gómez, *Classifying smooth supersolutions for a general class of elliptic boundary value problems*, Adv. Differential Equations **8** (2003), no. 9, 1025–1042. MR1989288 (2005b:35054)

[19] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Academia, Prague, 1967. MR0227584 (37 #3168)

[20] N. S. Trudinger, *Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients*, Math. Z. **156** (1977), no. 3, 291–301. MR0470460 (57 #10214)

[21] M. Warma, *The Robin and Wentzell-Robin Laplacians on Lipschitz domains*, Semigroup Forum **73** (2006), no. 1, 10–30. MR2277314 (2007g:35032)

Daniel Daners
School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia
e-mail: `D.Daners@maths.usyd.edu.au`
URL: `www.maths.usyd.edu.au/u/daners/`