AN ALTERNATIVE APPROACH TO THE FABER-KRAHN INEQUALITY FOR ROBIN PROBLEMS

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Abstract. We give a simple proof of the Faber-Krahn inequality for the first eigenvalue of the p-Laplace operator with Robin boundary conditions. The techniques introduced allow to work with much less regular domains by using test function arguments. We substantially simplify earlier proofs, and establish the sharpness of the inequality for a larger class of domains at the same time.

1. Introduction

If Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < \infty$ and $\beta > 0$, then it is well known that

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2} u && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p-2} u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

has a first eigenvalue $\lambda_1(\Omega)$. Here Δ_p is the p-Laplacian given by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. That eigenvalue is given by

$$\lambda_1(\Omega) = \min_{\substack{u \in W_p^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^p \, dx + \beta \int_{\partial\Omega} |u|^p \, d\sigma}{\int_\Omega |u|^p \, dx}. \tag{1.2}$$

It is isolated and simple, and the corresponding eigenfunction can be chosen to be positive. The aim of this paper is twofold. First, we extend results from [8, 10] and [6] to a larger class of domains. Second, we substantially simplify many arguments. The idea is to work only with the weak form of the equation, and replace most key arguments requiring boundary regularity by test function arguments. The main result is the following isoperimetric inequality.

Date: February, 2009, to appear in Calculus of Variations and Partial Differential Equations, Springer Verlag, doi:10.1007/s00526-009-0252-3.

2000 *Mathematics Subject Classification.* 35P15, (35J25, 35J60).

This work was done during a visit of D. Bucur to the University of Sydney supported by a grant of the Australian Research Council.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded Lipschitz domain. If B is a ball of the same volume as Ω, then $\lambda_1(B) \leq \lambda_1(\Omega)$ with equality if and only if Ω is a ball.*

Apart from a simpler proof, the improvement over [6, 10] is the uniqueness in the class of Lipschitz domains rather than just C^2-domains. For more information on the history and background of the problem we refer to [8].

We expect that our new method is powerful enough to deal with an even larger class of domains, provided a suitable weak formulation for the eigenvalue problem and good trace theorems are used. We refer to Section 6 for a detailed discussion of the issues involved. The result may also be useful to improve the isoperimetric inequality for the second eigenvalue given in [14] in case of $p = 2$.

2. A level set representation for the first eigenvalue

The aim of this section is to use a test function argument to establish the level set representation for the first eigenvalue $\lambda_1(\Omega)$ of (1.1) from [8, Proposition 2.1] and [6, Proposition 3.1]. As a consequence we do not need certain smoothness assumptions on the domain, and the eigenfunction does not need to be as regular as in the above mentioned references. In particular, in case of the p-Laplacian, we can avoid the rather technical regularisation procedure used in [6, Section 3].

The eigenvalue problem is understood in the weak sense, that is, λ is an eigenvalue if there exists a non-zero function $u \in W_p^1(\Omega)$, called an eigenfunction, such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta |u|^{p-2} uv \, d\sigma = \lambda \int_\Omega |u|^{p-2} uv \, dx \qquad (2.1)$$

for all $v \in W_p^1(\Omega)$. It is well known that (1.1) has an isolated first eigenvalue. The corresponding eigenfunction ψ is simple and can be chosen such that $\psi(x) > 0$ for all $x \in \Omega$ and normalised such that $\|\psi\|_\infty = 1$. The proofs in [17] using the direct method in the calculus of variations are easily adapted to our situation (see [6, Section 2] or [16, Theorem 3.4]). Note that the very elegant and simple proof from [13, 2] could be adapted.

From standard regularity theory $\psi \in C^1(\Omega)$ (see [21]). We also use that the eigenfunction is continuous up to the boundary, even though we believe that this is not really necessary if good trace theorems are available. For a discussion of that see Section 6.

Lemma 2.1. *If Ω is Lipschitz and ψ is the first eigenfunction of* (1.1)*, then $\psi \in C(\bar{\Omega}) \cap C^1(\Omega)$.*

Proof. We have seen already that $\psi \in C^1(\Omega)$. For the boundary regularity note that $\psi \in L_\infty(\Omega)$. (see [9, Theorem 2.7 and Section 4], where that fact is proved for arbitrary domains). Now the arguments in [15, page 466/467] imply that ψ is Hölder continuous on $\bar{\Omega}$. The aim of that reference is to prove that $\nabla\psi$ is Hölder continuous. An analysis of that proof shows that for the Hölder continuity of ψ only the Lipschitz continuity of Ω is needed. For the case $p = 2$ also see [22]. □

We next define a functional.

Definition 2.2. For $t \in (0, 1)$ we set

$$\begin{aligned} U_t &:= \{x \in \Omega \colon \psi(x) > t\}, \\ S_t &:= \{x \in \Omega \colon \psi(x) = t\}, \\ \Gamma_t &:= \{x \in \partial\Omega \colon \psi(x) > t\}. \end{aligned}$$

For a measurable function $\varphi \colon \Omega \to [0, \infty)$ we set

$$H_\Omega(U_t, \varphi) := \frac{1}{|U_t|}\Big(\int_{S_t} \varphi \, d\sigma + \int_{\Gamma_t} \beta \, d\sigma - (p-1)\int_{U_t} \varphi^{p'} \, dx\Big)$$

whenever the integral exists. Here σ is the $(N-1)$-dimensional Hausdorff measure and $|U_t|$ denotes the Lebesgue measure of the set U_t.

One of the main tools we use is the co-area formula asserting that for every non-negative measurable function v on Ω

$$\int_\Omega v|\nabla\psi| \, dx = \int_0^\infty \int_{S_t} v \, d\sigma \, dt. \tag{2.2}$$

For a proof we refer to [19, Section 1.2.4] or [12, Equation (3.4)]. We prove the following level set representation of the first eigenvalue.

Proposition 2.3. *Let $\psi > 0$ be the eigenfunction of* (1.1) *corresponding to $\lambda_1(\Omega)$. Then*

$$\lambda_1(\Omega) = H_\Omega(U_t, |\nabla\psi|^{p-1}/\psi^{p-1})$$

for almost all $t \in (0, 1)$.

Proof. Fix $t \in (0, 1)$ and let $\varepsilon \in (0, t)$. Define the function

$$\varphi_\varepsilon := \frac{1}{\psi^{p-1}} \min\Big\{1, \Big(\frac{\psi - t}{\varepsilon}\Big)^+\Big\}.$$

Then clearly φ_ε is increasing in ε and

$$\varphi_\varepsilon \to \frac{1}{\psi^{p-1}} 1_{U_t} \tag{2.3}$$

as $\varepsilon \to 0$. As usual, 1_{U_t} is the indicator function for the set U_t. Moreover, $\varphi_\varepsilon \in W_p^1(\Omega)$ and by [11, Section 4.2.2],

$$\nabla\varphi_\varepsilon = \begin{cases} -(p-1)\dfrac{\nabla\psi}{|\psi|^p} & \text{if } \psi > t+\varepsilon \text{ (on } U_{t+\varepsilon}\text{)},\\ \dfrac{1}{\varepsilon}\Big((p-1)\dfrac{t}{\psi} - p + 2\Big)\dfrac{\nabla\psi}{\psi^{p-1}} & \text{if } t < \psi < t+\varepsilon,\\ 0 & \text{otherwise.} \end{cases}$$

Note that if t is below the infimum of ψ, then φ_ε is simply given by $\psi^{-(p-1)}$. We now look at each term of (2.1) setting $u = \psi$ and $v = \varphi_\varepsilon$. For the first term we have

$$\begin{aligned}\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\varphi_\varepsilon\,dx = -(p-1)\int_{U_{t+\varepsilon}} \frac{|\nabla\psi|^p}{\psi^p}\,dx \\ + \frac{1}{\varepsilon}\int_{U_t\setminus U_{t+\varepsilon}} \Big((p-1)\frac{t}{\psi} - p + 2\Big)\frac{|\nabla\psi|^p}{\psi^{p-1}}\,dx\end{aligned}$$

for all $0 < \varepsilon < t$. We rewrite the second integral using the co-area formula (2.2). We then get

$$\begin{aligned}\int_{U_{t-\varepsilon}\setminus U_t} \Big((p-1)\frac{t}{\psi} - p + 2\Big)\frac{|\nabla\psi|^p}{\psi^{p-1}}\,dx \\ = \int_t^{t+\varepsilon} \Big((p-1)\frac{t}{\tau} - p + 2\Big)\int_{S_\tau} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma\,d\tau\end{aligned}$$

Since $\psi \in W_p^1(\Omega)$ the above also shows that the function

$$s \mapsto \int_t^s \Big((p-1)\frac{t}{\tau} - p + 2\Big)\int_{S_\tau} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma\,d\tau$$

is locally absolutely continuous on $(0,1)$. Using that such functions are differentiable almost everywhere (see [20, Theorem 8.17]) we get that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{U_t\setminus U_{t+\varepsilon}} \Big((p-1)\frac{t}{\psi} - p + 2\Big)\frac{|\nabla\psi|^p}{\psi^{p-1}}\,dx \\ &\qquad = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Big((p-1)\frac{t}{\tau} - p + 2\Big)\int_{S_\tau} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma\,d\tau \\ &\qquad \to \Big((p-1)\frac{t}{t} - p + 2\Big)\int_{S_t} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma = \int_{S_t} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma\end{aligned}$$

for almost all $t \in (0,1)$ as $\varepsilon \to 0$. Looking at the other terms in (2.1) we get by using (2.3) and the monotone convergence theorem

$$\int_{\partial\Omega} \beta|\psi|^{p-2}\psi\varphi_\varepsilon\,d\sigma \to \int_{\Gamma_t} \beta\,d\sigma. \tag{2.4}$$

Similarly

$$\int_{\Omega} |\psi|^{p-2}\psi\varphi_\varepsilon\,dx \to \int_{U_t} 1\,dx = |U_t|$$

as $\varepsilon \to 0$. Hence by letting $\varepsilon \to 0$ in the identity

$$\int_{\Omega} |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\varphi_\varepsilon\,dx + \int_{\partial\Omega} \beta|\psi|^{p-2}\psi\varphi_\varepsilon\,d\sigma = \lambda_1(\Omega)\int_{\Omega} |\psi|^{p-2}\psi\varphi_\varepsilon\,dx$$

we get

$$-(p-1)\int_{U_t} \frac{|\nabla\psi|^p}{\psi^p}\,dx + \int_{S_t} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma + \int_{\Gamma_t} \beta\,d\sigma = \lambda_1(\Omega)|U_t|$$

for almost all $t \in (0,1)$. Rearranging, the assertion of the proposition follows. $\square$

3. An estimate for the first eigenvalue

We can use the representation in Proposition 2.3 to estimate the functional $H_\Omega(U_t,\varphi)$ in terms of $\lambda_1(\Omega)$.

Proposition 3.1. *Let $\varphi\colon \Omega \to [0,\infty)$ be a measurable function such that $\varphi \in L_{p'}(U_t)$ for all $t > 0$, where p' is such that $1/p + 1/p' = 1$. Set*

$$w := \varphi - \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}} \quad \text{and} \quad F(t) := \int_{U_t} w\frac{|\nabla\psi|}{\psi}\,dx.$$

Then $F\colon (0,1) \to \mathbb{R}$ is locally absolutely continuous and

$$H_\Omega(U_t,\varphi) \le \lambda_1(\Omega) - \frac{1}{|U_t|t^{p-1}}\frac{d}{dt}\big(t^pF(t)\big) \tag{3.1}$$

for almost all $t \in (0,1)$. Moreover, there is strict inequality in (3.1) if and only if $\varphi \neq |\nabla\psi|^{p-1}/\psi^{p-1}$ in U_t on a set of non-zero measure.

Proof. We start by proving an elementary inequality. Consider the function $g(\varphi) = \varphi^{p'}$ defined for $\varphi \ge 0$. Since $g''(\varphi) = p'(p'-1)\varphi^{p'-2} > 0$ for all $\varphi > 0$, the function g is strictly convex. The tangent of g at $v \ge 0$ is given by $v^{p'} + p'v^{p'-1}(\varphi - v)$, so by the strict convexity

$$\varphi^{p'} \ge v^{p'} + p'v^{p'-1}(\varphi - v) \tag{3.2}$$

for all $\varphi \geq 0$ with strict inequality if and only if $\varphi \neq v$. We next consider a new representation of $\lambda_1(\Omega)$. From the definitions of $H_\Omega(U_t, \varphi)$ and w as well as Proposition 2.3 we immediately get

$$H_\Omega(U_t, \varphi) = \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\,d\sigma - (p-1)\int_{U_t} \varphi^{p'} - \frac{|\nabla\psi|^p}{\psi^p}\,dx\Big).$$

Applying (3.2) with $v := |\nabla\psi|^{p-1}/\psi^{p-1}$ we get

$$\varphi^{p'} - \frac{|\nabla\psi|^p}{\psi^p} \geq \frac{p}{p-1} w \frac{|\nabla\psi|}{\psi} \tag{3.3}$$

with strict inequality if and only if $w \neq 0$. Therefore,

$$H_\Omega(U_t, \varphi) \leq \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\,d\sigma - p\int_{U_t} w\frac{|\nabla\psi|}{\psi}\,dx\Big)$$

for almost all $t \in (0,1)$. We next want to derive a different representation of the above volume integral. We note that

$$\frac{|\nabla\psi|}{\psi} \leq \frac{1}{t}|\nabla\psi|$$

on U_t if $t > 0$. Hence $|\nabla\psi|/\psi \in L_p(U_t)$ for all $t > 0$. By the co-area formula (2.2)

$$\int_{U_t} \frac{|\nabla\psi|^p}{\psi^p}\,dx = \int_t^1 \frac{1}{\tau}\int_{S_\tau} \frac{|\nabla\psi|^{p-1}}{\psi^{p-1}}\,d\sigma\,d\tau < \infty$$

and since $\varphi \in L_{p'}(U_t)$

$$\int_{U_t} \varphi\frac{|\nabla\psi|}{\psi}\,dx = \int_t^1 \frac{1}{\tau}\int_{S_\tau} \varphi\,d\sigma\,d\tau < \infty$$

for all $t \in (0,1)$. Because the integrands are non-negative this implies that the function

$$\Big(t \to \frac{1}{t}\int_{S_t} w\,d\sigma\Big) \in L_1((\delta,1)) \tag{3.4}$$

for all $\delta > 0$ and that

$$F(t) = \int_{U_t} w\frac{|\nabla\psi|}{\psi}\,dx = \int_t^1 \int_{S_\tau} \frac{w}{\psi}\,d\sigma\,d\tau = \int_t^1 \frac{1}{\tau}\int_{S_\tau} w\,d\sigma\,d\tau.$$

Therefore

$$H_\Omega(U_t, \varphi) \leq \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\,d\sigma - pF(t)\Big) \tag{3.5}$$

for all $t \in (0,1)$. By (3.4), the function F is absolutely continuous on $(\delta, 1)$ for all $\delta > 0$. Hence for almost all $t \in (0,1)$

$$-\frac{d}{dt}\big(t^p F(t)\big) = t^p\frac{1}{t}\int_{S_t} w\,d\sigma - pt^{p-1}F(t) = t^{p-1}\Big(\int_{S_t} w\,d\sigma - pF(t)\Big).$$

To get (3.1) we substitute the above into (3.5). □

We use the above tho prove the following theorem which generalises [10, Theorem 2.2] and [6, Theorem 4.2] to a larger class of domains. At the same time we relax the restrictions on the boundary behaviour of φ made in [4, 6, 8, 10]. We now get an estimate for $H_\Omega(U_t,\varphi)$ from below if $\varphi \neq |\nabla\psi|^{p-1}/\psi^{p-1}$. Note that the case $\varphi = |\nabla\psi|^{p-1}/\psi^{p-1}$ is covered in Proposition 2.3

Theorem 3.2. *Let $\varphi\colon \Omega \to [0,\infty)$ be a measurable function such that $\varphi \in L_{p'}(\Omega)$. Furthermore, suppose that $\varphi \neq |\nabla\psi|^{p-1}/\psi^{p-1}$ on a set of non-zero measure. Then there exists a set $S \subset (0,1)$ of non-zero measure such that*

$$\lambda_1(\Omega) > H_\Omega(U_t,\varphi) \tag{3.6}$$

for all $t \in S$.

Proof. We give a proof by contradiction, assuming that

$$\lambda_1(\Omega) \leq H_\Omega(U_t,\varphi)$$

for almost all $t \in (0,1)$. Then from Proposition 3.1 we have

$$\lambda_1(\Omega) \leq H_\Omega(U_t,\varphi) \leq \lambda_1(\Omega) - \frac{1}{|U_t|t^{p-1}}\frac{d}{dt}\big(t^pF(t)\big)$$

for almost all $t \in (0,1)$. Setting $G(t) := t^pF(t)$ we have in particular

$$G'(t) = \frac{d}{dt}\big(t^pF(t)\big) \leq 0$$

for almost all $t \in (0,1)$. Hence G is decreasing on $(0,1)$. Since $F(1) = 0$ we have $G(1) = 0$ and hence $G(t) \geq 0$ for all $t \in (0,1)$. By assumption $\varphi \neq |\nabla\psi|^{p-1}/\psi^{p-1}$ on a set of nonzero measure in Ω. Since the sets U_t exhaust Ω this is also true for U_t if t is small enough. Because (3.1) is strict in that case, $G'(t) < 0$ for $t > 0$ in a neighbourhood of zero and so

$$\lim_{t\to 0+} G(t) > 0.$$

We show that this is not possible. By definition of F and w and Hölder's inequality

$$\begin{aligned} F(t) = \int_{U_t} w\frac{|\nabla\psi|}{\psi}\,dx &= \int_{U_t} \varphi\frac{|\nabla\psi|}{\psi}\,dx - \int_{U_t}\frac{|\nabla\psi|^p}{\psi^p}\,dx \\ &\leq \int_{U_t} \varphi\frac{|\nabla\psi|}{\psi}\,dx \leq \frac{1}{t}\int_{U_t}\varphi|\nabla\psi|\,dx \leq \frac{1}{t}\|\varphi\|_{p'}\|\nabla\psi\|_p \end{aligned}$$

for all $t \in (0,1)$. Hence, since $p > 1$, we get

$$0 < \lim_{t\to 0+} G(t) = \lim_{t\to 0+} t^p F(t) \le t^{p-1} \|\varphi\|_{p'} \|\nabla\psi\|_p = 0$$

which is a contradiction. □

4. The eigenvalue problem on balls

In this section we prove some results on the first eigenvalue and the corresponding eigenfunction if Ω is a ball B_r of radius $r > 0$. We assume for simplicity that B_r is centred at zero. In the spirit of this paper we prove these results in a natural fashion from the variational characterisation (1.2) of $\lambda_1(B_r)$, replacing an explicit representation using Bessel functions in [8], and a cumbersome differential inequality in [6, Proposition 2.8]. We start by showing that $\lambda_1(B_r)$ is a strictly decreasing function of r.

Lemma 4.1. *The function $r \mapsto \lambda_1(B_r)$ is strictly decreasing on $(0,\infty)$.*

Proof. Let $0 < r < R$ and suppose ψ is the eigenfunction on B_r corresponding to $\lambda_1(B_r)$. Setting $t := r/R$ we define $v(x) := \psi(tx)$ for $x \in B_R$. Then $\nabla v(x) = t\nabla\psi(tx)$, so by (1.2) and the transformation formula

$$\begin{aligned}\lambda_1(B_R) &\le \frac{\int_{B_R} |\nabla v|^p \,dx + \beta \int_{\partial B_R} |v|^p \,d\sigma}{\int_{B_R} |v|^p \,dx} \\ &= \frac{t^p \int_{B_r} |\nabla \psi|^p \,dx + t\beta \int_{\partial B_r} |\psi|^p \,d\sigma}{\int_{B_r} |\psi|^p \,dx} \\ &< \frac{\int_{B_r} |\nabla \psi|^p \,dx + \beta \int_{\partial B_r} |\psi|^p \,d\sigma}{\int_{B_r} |\psi|^p \,dx} = \lambda_1(B_r),\end{aligned}$$

where we have used that $t < 1$ in the last inequality. □

If ψ is the eigenfunction on B_R corresponding to $\lambda_1(B_R)$, then the simplicity implies that ψ is radially symmetric. In case of $p = 2$ the eigenfunction is positive and strictly decreasing in the radial direction. The same turns out to be true for all $p \in (1,\infty)$ (see [6, Proposition 2.7]).

In particular, $\nabla\psi \neq 0$ on $B_R \setminus \{0\}$, so in that domain the operator Δ_p is uniformly elliptic. Standard regularity theory for linear equations with diffusion coefficient $|\nabla u|^{p-2}$ implies that $\psi \in C^\infty(B_R \setminus \{0\})$. Because of the radial symmetry of the eigenfunction also

$$\beta_r := \frac{|\nabla\psi(x)|^{p-1}}{\psi(x)^{p-1}}$$

is constant for $|x| = r$. The following result is essential for our proof of Theorem 1.1. Its proof is much simpler and more natural than the one in [6, Proposition 2.8].

Proposition 4.2. *The function $r \mapsto \beta_r$ is strictly increasing on $[0, R]$ with $\beta_0 = 0$ and $\beta_R = \beta$.*

Proof. Note that $\beta_0 = 0$ since $\nabla\psi(0) = 0$, and $\beta_R = \beta$ by the boundary condition. Hence look at $r \in (0, R)$. We know already that ψ is smooth and strictly decreasing in the radial direction, so the outer unit normal to B_r is in the direction of $-\nabla\psi$. Hence $\lambda_1(B_R)$ and ψ satisfies the eigenvalue problem

$$\begin{aligned} -\Delta_p\psi &= \lambda_1(B_R)|\psi|^{p-2}\psi && \text{in } B_r,\\ |\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu} + \beta_r|\psi|^{p-2}\psi &= 0 && \text{on } \partial B_r \end{aligned}$$

for every $r \in (0, R)$. Denote the first eigenvalue on Ω with boundary coefficient β by $\lambda_1(\Omega, \beta)$. Since the first eigenvalue is the only eigenvalue with a positive eigenfunction, we conclude that

$$\lambda(B_r, \beta_r) = \lambda(B_R)$$

for all $r \in (0, R]$. Hence by Lemma 4.1 and (1.2) we get for $0 < r_1 < r_2 \leq R$

$$\begin{aligned} \frac{\int_{B_{r_1}} |\nabla\psi|^p\,dx + \beta_{r_1}\int_{\partial B_{r_1}} |\psi|^p\,d\sigma}{\int_{B_{r_1}} |\psi|^p\,dx} &= \lambda_1(B_{r_1}, \beta_{r_1}) = \lambda_1(B_{r_2}, \beta_{r_2})\\ &< \lambda_1(B_{r_1}, \beta_{r_2}) \leq \frac{\int_{B_{r_1}} |\nabla\psi|^p\,dx + \beta_{r_2}\int_{\partial B_{r_1}} |\psi|^p\,d\sigma}{\int_{B_{r_1}} |\psi|^p\,dx}, \end{aligned}$$

which is only possible if $\beta_{r_1} < \beta_{r_2}$. ☐

5. Proof of the isoperimetric inequality

The aim of this section is to prove Theorem 1.1. We assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set and let $B \subset \mathbb{R}^N$ the ball of radius $R > 0$ centred at zero having the same volume as Ω. We first construct a suitable function φ on Ω by rearranging $|\nabla\psi_*|^{p-1}/\psi_*^{p-1}$, where ψ_* is a positive first eigenfunction of (1.1) for $\Omega = B$. We know from Section 4 that ψ_* is radially symmetric and so

$$\beta_r := \varphi_*(x) := \frac{|\nabla\psi_*(x)|^{p-1}}{\psi_*^{p-1}(x)}$$

for $|x| = r$ is well defined for $r \in [0, R]$. As in earlier sections we let ψ be the positive eigenfunction of (1.1) with $\|\psi\|_\infty = 1$. Given $t \in (0, 1)$ we define $r(t)$ to be radius the ball $B_{r(t)}$ centred at zero having the same volume as U_t, where U_t is the level set of ψ as in Definition 2.2. Then we define

$$\varphi(x) := \beta_{r(t)}$$

whenever $x \in \Omega$ and $\psi(x) = t$.

Lemma 5.1. *The function $\varphi\colon \Omega \to \mathbb{R}$ defined above is measurable and $0 \leq \varphi(x) < \beta$ for all $x \in \Omega$.*

Proof. By the continuity of ψ we have $|U_s| < |U_t|$ if $0 \leq t < s \leq 1$. Hence $t \mapsto r(t)$ is strictly increasing. By Proposition 4.2 we conclude that $t \mapsto \beta_{r(t)}$ is strictly increasing with maximum β for $t = 1$. This implies that

$$\{x \in \Omega\colon \psi(x) \geq t\} = \{x \in \Omega\colon \varphi(x) \leq r(t)\}$$

is closed for all $t \in (0, 1)$, and so φ is bounded and measurable. Since $0 \leq \varphi_*(x) < \beta$ for all $x \in B$ we also have $0 \leq \varphi(x) < \beta$ for all $x \in \Omega$. □

We also need to relate S_t and Γ_t to the boundary of U_t, so we can apply the geometric isoperimetric inequality.

Lemma 5.2. *There exists an at almost countable set $Q \subset (0, 1)$ such that $\sigma(\partial U_t) \leq \sigma(S_t) + \sigma(\Gamma_t)$ for all $t \in (0, 1) \setminus Q$.*

Proof. Since $\psi \in C(\Omega)$ it is clear that $\partial U_t \cap \Omega \subset S_t$ for all $t > 0$. Similarly, since $\psi \in C(\bar{\Omega})$ by Lemma 2.1

$$\partial U_t \cap \partial\Omega \subset \tilde{\Gamma}_t := \{x \in \partial\Omega\colon \psi(x) \geq t\}.$$

Now by [19, Section 1.2.3] we have

$$\sigma(\partial\Omega) = \int_{\partial\Omega} 1 \, d\sigma = \int_0^\infty \sigma(\Gamma_t) \, dt = \int_0^\infty \sigma(\tilde{\Gamma}_t) \, dt < \infty.$$

Since $\sigma(\Gamma_t) \leq \sigma(\tilde{\Gamma}_t)$ for all $t > 0$ the above implies that $\sigma(\Gamma_t) = \sigma(\tilde{\Gamma}_t)$ for almost all $t \geq 0$. Moreover, the functions $t \mapsto \sigma(\Gamma_t)$ and $t \mapsto \sigma(\tilde{\Gamma}_t)$ are monotone and therefore continuous except for an at most countable set Q. Hence

$$\sigma(\partial U_t) = \sigma(\partial U_t \cap \Omega) + \sigma(\partial U_t \cap \partial\Omega) \leq \sigma(S_t) + \sigma(\Gamma_t)$$

for all $t \in (0, 1) \setminus Q$. □

Next we compare the functionals H_Ω and H_B.

Proposition 5.3. *Let φ be as defined above. Then*

$$H_\Omega(U_t, \varphi) \geq H_B(B_{r(t)}, \varphi_*) = \lambda_1(B)$$

for all $t \in (0,1) \setminus Q$, where Q is the set from Lemma 5.2. Moreover, there is equality if and only if $\sigma(\Gamma_t) = 0$ and U_t is a ball except possibly for a set of measure zero.

Proof. Since $\varphi(x) = \beta_{r(t)}$ on S_t by definition, the isoperimetric inequality (see [19, Theorem 6.1.6 and Remark 6.2.2]) and Lemma 5.2 imply

$$\begin{aligned}\int_{\partial B_{r(t)}} \varphi_* \, d\sigma &= \beta_{r(t)}\sigma(\partial B_{r(t)}) \leq \beta_{r(t)}\sigma(\partial U_t) \\ &\leq \beta_{r(t)}\sigma(S_t) + \beta_{r(t)}\sigma(\Gamma_t) \leq \int_{S_t} \varphi \, d\sigma + \int_{\Gamma_t} \beta \, d\sigma \qquad (5.1)\end{aligned}$$

for all $t \in (0,1)$, where for the last inequality we used that $\varphi \leq \beta$. There is clearly equality if U_t is a ball and $\sigma(\Gamma_t) = 0$. For the converse note that $\beta_{r(t)} < \beta$ for all $r \in (0, R)$ by Proposition 4.2. Hence the last inequality in (5.1) is strict unless $\sigma(\Gamma_t) = 0$. If U_t does not have finite perimeter, then U_t is not a ball and by [19, Remark 6.2.2] the first inequality in (5.1) is strict. If U_t has finite perimeter, then by the sharpness of the isoperimetric inequality (see [18, Theorem 3.1]) the first inequality in (5.1) is strict unless U_t is a ball up to a set of measure zero. Finally, by definition $|U_t| = |B_{r(t)}|$ for all $t \in (0,1)$ and so by [19, Section 1.2.3]

$$\int_\Omega \varphi^{p'} \, dx = \int_{B_{r(t)}} \varphi_*^{p'} \, dx.$$

The assertion of the proposition now follows by using Definition 2.2. ☐

For the sharpness of the inequality we need the following lemma.

Lemma 5.4. *Suppose that $\lambda_1(\Omega) = \lambda_1(B)$. Then*

$$H_\Omega(U_t, \varphi) = H_B(B_{r(t)}, \varphi_*) = \lambda_1(B)$$

and for almost all $t \in (0,1)$, U_t is a ball except possibly for a set of zero $(N-1)$-dimensional Hausdorff measure.

Proof. Suppose that $\varphi \neq |\nabla\psi|^{p-1}/\psi^{p-1}$ on a set of positive measure. Then Theorem 3.2 implies that there exist $S \subset (0,1)$ of positive measure such that $\lambda_t(\Omega) > H_\Omega(U_t, \varphi)$ for almost all $t \in S$. Hence by Proposition 2.3

$$\lambda_1(B) = \lambda_1(\Omega) > H_\Omega(U_t, \varphi) \geq H_B(B_{r(t)}, \varphi_*) = \lambda_1(B)$$

for all $t \in S$. Since this is a contradiction $\varphi = |\nabla\psi|^{p-1}/\psi^{p-1}$ almost everywhere. By Proposition 5.3 U_t is a ball up a set of zero $(N-1)$-dimensional Hausdorff measure. □

We are finally in a position to complete the proof of Theorem 1.1. First observe that Proposition 2.3 and Theorem 3.2 imply the existence of $t \in (0,t)$ such that $\lambda_1(\Omega) \geq H_\Omega(U_t,\varphi)$. Hence $\lambda_1(\Omega) \geq \lambda_1(B)$ by Proposition 5.3. For the sharpness of the inequality assume that $\lambda_1(\Omega) = \lambda_1(B)$. Then by Lemma 5.4 U_t is a ball for almost all $t \in U_t$. Since U_t, $t \in (0,1)$, are nested sets it follows that $\Omega = \bigcup_{t\in(0,1)} U_t$ is a ball except possibly a set of measure zero. Since Ω is Lipschitz, it needs to be a ball.

6. Remarks on non-smooth domains

Our main regularity assumption on Ω, namely the Lipschitz character, could be weakened. First of all we have to make sure is that Γ_t as given in Definition 2.2 makes sense. For that we need to assume that the Robin problem is well posed in a function space where a suitable trace can be defined on $\partial\Omega$. Then, one can repeat the arguments in the proof of Proposition 2.3 as soon as the (local) trace of ψ can be related to the trace of $1/\psi$.

A second key argument is the application of the geometric isoperimetric inequality in the proof of Proposition 5.3. For that we need to relate Γ_t and S_t to the boundary of U_t. In fact we only need that

$$P(U_t) = \sigma(\partial^* U_t) \leq \sigma(\Gamma_t) + \sigma(S_t)$$

for almost all $t > 0$, where $P(U_t)$ denotes the perimeter and $\partial_t^* U_t$ the reduced boundary of U_t as defined for instance in [11, Chapter 5]. Such an inequality is valid if

$$\partial^* U_t \subset \Gamma_t \cup S_t \tag{6.1}$$

except for a set of zero $(N-1)$-dimensional Hausdorff measure.

For the sharpness of the inequality we also get more information from the proof of Proposition 5.3. Assuming equality we conclude that

$$\sigma(\partial U_t) = \sigma(S_t) = \sigma(B_{r(t)})$$

and $\sigma(\Gamma_t) = 0$. Hence U_t is not just a ball up to a set of measure zero, but a set of $(N-1)$-dimensional Hausdorff measure zero. This gives us additional information in case of non-smooth domains.

If Ω is an open set, not necessarily smooth, [7] used the space introduced by Maz'ja in [19, Section 4.11.6] to introduce a weak framework for the Robin problem. As it was noticed in [1], the trace operator in this space may not be well defined for sets which have a boundary containing a very large number of points with zero density with respect to Ω. However, even then there is hope that our method works by setting the trace to zero on those parts of the boundary on which it is not well defined. We then would need to try to show that (6.1) holds. In particular we would need to show that $\partial^* U_t \cap \partial\Omega$ is contained in the part of $\partial\Omega$ where the trace is well defined. An indication that this could be true is the example given in [1, Example 4.2]. Hence it seems evident that the right way to deal with the trace problem should involve tools of geometric measure theory, the reduced boundary playing a fundamental role.

We expect that a sufficient condition in order to have a "good" trace property is that the $(N-1)$-dimensional Hausdorff measure restricted to the boundary is absolutely continuous with respect to the relative capacity as introduced in [1, 3]. From this point of view, the Lipschitz regularity could certainly be weakened at least to open sets for which the trace is locally defined up to a set of $(N-1)$-Hausdorff measure zero in a usual sense such as for instance domains with cusps.

A different approach to define pointwise traces up to sets of zero $(N-1)$-dimensional Hausdorff measure is established in [5], where the "natural" space for the weak formulation of the Robin problem relies on a particular class of functions of bounded variation.

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