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A Faber-Krahn inequality for Robin problems in any space dimension

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Abstract We prove a Faber-Krahn inequality for the first eigenvalue of the Laplacian with Robin boundary conditions, asserting that amongst all Lipschitz domains of fixed volume, the ball has the smallest first eigenvalue. We prove the result in all space dimensions using ideas from [M.-H. Bossel, C. R. Acad. Sci. Paris Sér. I Math. **302** (1986), 47–50], where a proof for smooth domains in the plane was given. The method does not involve the use of symmetrisation arguments. The results also imply variants of the Cheeger inequality for the first eigenvalue.

1 Introduction

The purpose of this paper is to prove an isoperimetric inequality similar to the well known Faber-Krahn inequality for the first eigenvalue of the Robin problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, with outer unit normal ν and $\beta > 0$ constant. It is well known that the eigenvalues of (1.1) form a sequence $0 < \lambda_1 < \lambda_2 < \dots$. The eigenvalue $\lambda_1 = \lambda_1(\Omega)$ is referred to as the first eigenvalue

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of (1.1) on the domain Ω. In the model of a vibrating membrane, the boundary conditions describe an elastically supported membrane, and $\lambda_1(\Omega)$ is the ground frequency.

We shall prove that amongst all domains of fixed volume, the one on the ball has the smallest first eigenvalue. In the model of the membrane, this means that amongst all membranes of the same volume, the one on the ball has the lowest ground frequency. More precisely, the aim of the paper is to prove the following theorem.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded Lipschitz domain, $\beta > 0$ a constant and $\lambda_1(\Omega)$ the first eigenvalue of (1.1). If B is an open ball of the same measure as Ω, then $\lambda_1(B) \leq \lambda_1(\Omega)$.*

For the case $\beta = \infty$, that is, the Dirichlet problem (fixed membrane), this is the celebrated Faber-Krahn inequality (see for instance [4, 14, 21, 22, 27, 28, 30]). If $\beta = 0$, then $\lambda_1(\Omega) = 0$ for all bounded domains Ω, so the question we ask is not very interesting and other questions become relevant (see for instance [4, 18, 27]). If $\beta \in (0,\infty)$, it is known that there exists a positive lower bound for $\lambda_1(\Omega)$. In [10, Corollary 2.4] a lower bound in terms of β, the measure of Ω and the isoperimetric constant is given. In [23, 29], it is shown that if B is a ball circumscribing Ω, then $\lambda_1(B) \leq \lambda_1(\Omega)$. Note that this lower bound is not evident as, unlike in case of the Dirichlet problem, $\lambda_1(\Omega)$ is not a monotone function of Ω (see [29] for a counter example). Other isoperimetric inequalities for Robin problems and a comparison between various lower bounds for $\lambda_1(\Omega)$ are given in [32–34].

For domains in the plane Theorem 1.1 has been proved in [6] (see also [5, 7]). For $N \geq 3$ Theorem 1.1 has been open for many years. It has been taken up into a recent list of open isoperimetric problems (see [18, Problem 14] or [10]). The difficulty in proving Theorem 1.1 is that the level curves (or surfaces in case $N \geq 3$) of the first eigenfunction of (1.1) are not closed in Ω, so the usual symmetrisation techniques to decrease the Dirichlet integral do not seem to be useful. Symmetrisation is avoided in [6] by using an alternative estimate of $\lambda_1(\Omega)$.

We generalise the approach in [6] and prove the desired inequality in any space dimension for a larger class of domains. At the same time we rectify some shortcomings in [6] and fill in essential technical details not provided there. In addition to that we set up the proof such that it works simultaneously for Dirichlet boundary conditions.

Our proof proceeds in several steps. In Section 2 we introduce a functional of the level sets of the first eigenfunction of (1.1) and derive an alternative representation of that functional. The idea to use that functional originates from a conformal invariant called extremal length due to Ahlfors and Beurling [2, Chapter 4] and results by Hersch [19] as explained in [5]. The next step, presented in Section 3, is to establish an estimate for $\lambda_1(\Omega)$ in terms of that functional. All these results are valid with essentially the same proof for mixed Dirichlet-Robin problems with $\beta \geq 0$ non-constant. We note that this estimate (from Theorem 3.1 and 3.5) can be used to prove variants of the *Cheeger inequality* in a similar manner as carried out in [6]. The next step is a proof of Theorem 1.1 for domains of class C^2 making use of properties of the radially symmetric problem on the ball. The final step consists of an approximation argument involving smoothing the domain, and then using the results on the continuous dependence of $\lambda_1(\Omega)$ on the domain from [9].

We suspect that the assertion of Theorem 1.1 remains true for *arbitrary* domains in the setting of [10] (see also [3]). In the extreme case of a domain with a fractal boundary the result is true. Indeed, in that case (1.1) degenerates into a Dirichlet problem (see [10, Remark 3.5(b)]). By the usual Faber-Krahn inequality for the Dirichlet problem we have $\lambda_1(\Omega) \geq \mu_1(B)$, where $\mu_1(B)$ is the first eigenvalue of the Dirichlet problem on the ball B with the same volume as Ω. Since $\lambda_1(B) \leq \mu_1(B)$ it follows that $\lambda_1(B) \leq \lambda_1(\Omega)$.

One question we leave open is that of the uniqueness of the minimising domain. More precisely, if $\lambda_1(B) = \lambda_1(\Omega)$ for a sufficiently smooth domain Ω with the same measure as the ball B, does it follow that Ω is a ball? For the Dirichlet problem, the question is answered in the affirmative, but the proof is not straightforward (see the discussion in [21, Section II.8]). The results in the present paper, in particular Remark 4.3, provide a strong indication that the methods can be used to derive a uniqueness result for Robin as well as Dirichlet boundary conditions. This will be attempted elsewhere.

2 A functional of the level sets

In this section we introduce a functional used to prove Theorem 1.1. Since the proof is essentially the same as for pure Robin boundary conditions, we will cover a more general situation than actually needed. We consider the mixed boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 && \text{on } \Gamma_1, \end{aligned} \tag{2.1}$$

on the C^2-domain Ω, where Γ_0, Γ_1 are disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$. We further assume that $\beta \in C^1(\Gamma_1)$, with $\beta(x) > 0$ for all $x \in \Gamma_1$. If $\Gamma_0 = \emptyset$, then we have a pure Robin problem as in (1.1) and if $\Gamma_1 = \emptyset$ we have the Dirichlet problem.

For open sets $U \subset \Omega$ we define the interior and exterior boundary by

$$\partial_i U := \partial U \cap \Omega \qquad \text{and} \qquad \partial_e U := \partial U \cap \partial\Omega.$$

Then clearly $\partial U = \partial_i U \cup \partial_e U$ is a disjoint union. Given $U \subset \Omega$ open such that $\overline{U} \cap \Gamma_0 = \emptyset$ and $\varphi \in C(\Omega)$ non-negative we define the functional

$$H_\Omega(U,\varphi) := \frac{1}{|U|}\Big(\int_{\partial_i U} \varphi\, d\sigma + \int_{\partial_e U} \beta\, d\sigma - \int_U |\varphi|^2\, dx\Big), \tag{2.2}$$

where σ denotes the $(N-1)$-dimensional Hausdorff measure on ∂U and $|U|$ the Lebesgue measure of U. If the boundary of U is Lipschitz, then σ is the usual surface measure (see [13, Section 3.3.2]). Since φ is continuous on $\partial_i U$ it follows that all integrals in $H_\Omega(U,\varphi)$ are defined. Note however that the first and last integrals appearing in $H_\Omega(U,\varphi)$ do not need to be finite for every choice of U and φ.

We next look at a family of particular sub-domains U, namely the level sets of the first eigenfunction ψ of (2.1). It is well known that

$$\psi \in W_p^2(\Omega) \cap C^\infty(\Omega) \tag{2.3}$$

for all $p \in (1,\infty)$ (see [1, Theorem 4.2] and standard results on interior regularity). By well known embedding theorems $\psi \in C^1(\overline{\Omega})$ (see [17, Corollary 7.11]). Moreover, ψ can be chosen such that $0 \le \psi(x)$ for all $x \in \overline{\Omega}$ and $\|\psi\|_\infty = 1$. We let

$$m := \min_{x\in\overline{\Omega}} \psi(x).$$

By the Hopf boundary maximum principle $\psi(x) > 0$ for all $x \in \Omega \cup \Gamma_1$ and ψ attains its minimum on $\partial\Omega$ (see [16, Theorem 2.15]). If $\Gamma_0 = \emptyset$, then $m > 0$ and otherwise $m = 0$. We now consider the level sets

$$U_t := \{x \in \Omega\colon\ \psi(x) > t\}. \tag{2.4}$$

By the continuity of ψ in Ω, the sets U_t are open. Note that the interior boundary of U_t is the level surface

$$S_t := \partial_i U_t = \{x \in \Omega\colon\ \psi(x) = t\} \tag{2.5}$$

and that $S_t = \emptyset$ if $t \notin (m,1]$. Note also that $\overline{U}_t \cap \Gamma_0 = \emptyset$ for all $t \in (m,1)$.

The main aim of this section is to establish a new representation of $H_\Omega(U_t,\varphi)$. Key to get the representation is the following result. The proof involves regularity properties of the level sets U_t and S_t which will be proved in Lemma 2.3 at the end of this section.

Proposition 2.1 *If ψ is a positive first eigenfunction of* (2.1), *then*

$$H_\Omega(U_t, |\operatorname{grad}\psi|/\psi) = \lambda_1(\Omega) \tag{2.6}$$

for almost all $t \in (m,1)$.

Proof From Lemma 2.3 given below we know that U_t is Lipschitz and S_t is of class C^∞ for almost all $t \in (m,1)$. We now fix $t \in (m,1)$ with these properties and prove (2.6) for that choice of t. Since ψ satisfies the boundary conditions in (2.1) we have

$$\beta = -\frac{1}{\psi}\frac{\partial \psi}{\partial \nu}$$

on Γ_1. Moreover, since S_t is a smooth level surface of ψ we also have

$$|\operatorname{grad}\psi| = -\frac{\partial \psi}{\partial \nu}$$

on S_t if ν denotes the outward pointing unit normal to U_t. Combining the two and using that $\Gamma_0 \cap \partial U_t = \emptyset$,

$$\int_{S_t} \frac{|\operatorname{grad}\psi|}{\psi}\,d\sigma + \int_{\partial_e U_t} \beta\,d\sigma = -\int_{\partial U_t} \frac{1}{\psi}\frac{\partial \psi}{\partial \nu}\,d\sigma.$$

We have chosen t such that U_t is Lipschitz. Since $\psi \in W_p^2(\Omega)$ for all $p \in (1,\infty)$ and $\psi \geq t > 0$ on $\overline{U}_t$ we have $|\operatorname{grad}\psi|/\psi \in W_p^1(U_t)$ for $p \in (1,\infty)$. Hence the divergence theorem applies (see [26, Théorème 3.1.1]) and thus

$$\begin{aligned}\int_{S_t} \frac{|\operatorname{grad}\psi|}{\psi}\,d\sigma + \int_{\partial_e U_t} \beta\,d\sigma &= -\int_{U_t} \operatorname{div}\Big(\frac{\operatorname{grad}\psi}{\psi}\Big)\,dx\\ &= -\int_{U_t} \frac{\Delta\psi}{\psi} - \frac{|\operatorname{grad}\psi|^2}{\psi^2}\,dx = \lambda_1(\Omega)|U_t| + \int_{U_t} \frac{|\operatorname{grad}\psi|^2}{\psi^2}\,dx.\end{aligned}$$

For the last equality we used that ψ is an eigenfunction of (2.1). If we substitute the above into the definition of $H_\Omega(U_t, |\operatorname{grad}\psi|/\psi)$ the assertion of the proposition follows. □

Theorem 2.2 *Let $\varphi \in C(\Omega)$ be non-negative and set*

$$w := \varphi - \frac{|\operatorname{grad}\psi|}{\psi}.$$

Then

$$H_\Omega(U_t,\varphi) = \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\,d\sigma - 2\int_t^1 \frac{1}{\tau}\int_{S_\tau} w\,d\sigma\,d\tau - \int_{U_t} |w|^2\,dx\Big) \tag{2.7}$$

for almost all $t \in (m,1)$.

Proof Note that

$$|\varphi|^2 = \Big(w + \frac{|\operatorname{grad}\psi|}{\psi}\Big)^2 = |w|^2 + 2\frac{w}{\psi}|\operatorname{grad}\psi| + \frac{|\operatorname{grad}\psi|^2}{\psi^2}.$$

Using the coarea formula (see [13, Section 3.4.3] or [24, Section 1.2.4]) and noting that $\psi = \tau$ on S_τ

$$\int_{U_t} \frac{w}{\psi} |\operatorname{grad} \psi| \, dx = \int_t^1 \int_{S_\tau} \frac{w}{\psi} \, d\sigma \, d\tau = \int_t^1 \frac{1}{\tau} \int_{S_\tau} w \, d\sigma \, d\tau.$$

The coarea formula applies to any non-negative measurable (not necessarily integrable) function, so the above works for any $\varphi \in C(\Omega)$. If we substitute the above into (2.2) and use (2.6), then the assertion of the theorem follows. □

We finally prove the properties of the level sets U_t used in the proof of Proposition 2.1. The properties are not completely obvious, and implicitly used in [6] without proof. Some alternative results are mentioned in Remark 2.4 after the proof of the lemma. Since Ω is of class C^2, the function ψ has an extension $\tilde{\psi} \in W_p^2(\mathbb{R}^N)$ with compact support (see [17, Theorem 7.25]). Choosing $p > N$ it follows from standard embedding theorems that $\tilde{\psi} \in C^1(\mathbb{R}^N)$. As usual we denote by $B(z, \delta)$ the open ball of radius δ centred at z.

Lemma 2.3 *If ψ, U_t and S_t are as above, then the following assertions hold.*

(1) *The function $t \mapsto \sigma(S_t)$ is in $L_1((0,1))$.*
(2) *The surfaces S_t are of class C^∞ and U_t is Lipschitz for almost all $t \in (m,1)$.*
(3) *There exist $c > 0$ and $t_1 \in (m,1)$ such that $\sigma(S_t) \le c\,\sigma(\partial\Omega)$ for all $t \in (m,t_1]$.*

Proof (1) By the coarea formula (see [13, Section 3.4.2] or [24, Section 1.2.4]) and (2.3)

$$\int_0^1 \sigma(S_t) \, dt = \|\operatorname{grad} \psi\|_1 < \infty.$$

Hence the function $t \mapsto \sigma(S_t)$ is in $L_1((0,1))$.

(2) As $\psi \in C^\infty(\Omega)$ Sard's lemma (see [20, Theorem 3.1.3]) implies that S_t is of class C^∞ for almost all $t \in (0,1)$. Since Ω is of class C^2 it is therefore sufficient to show that U_t can locally be represented by the graph of a Lipschitz function where the level surfaces S_t meet Γ_1, that is, near $\overline{S}_t \cap \Gamma_1$ (note that $\overline{S}_t \cap \Gamma_0 = \emptyset$). Let $\tilde{\psi} \in C^1(\mathbb{R}^N)$ be an extension of ψ as explained above. Fix $x_0 \in \overline{S}_t \cap \Gamma_1$ and $t \in (m,1)$. We know that $\psi > 0$ on Γ_1, so by the boundary conditions and the assumption that $\beta(x_0) > 0$

$$\operatorname{grad} \tilde{\psi}(x_0) \cdot \nu(x_0) = -\beta \psi(x_0) < 0. \tag{2.8}$$

In particular $\operatorname{grad} \tilde{\psi}(x_0) \neq 0$. Hence by the implicit function theorem $\tilde{S}_t := \{x \in \mathbb{R}^N : \tilde{\psi}(x) = t\}$ is locally near x_0 a C^1-hypersurface. Moreover, $\tilde{S}_t = \partial \tilde{U}_t$, where $\tilde{U}_t := \{x \in \mathbb{R}^N : \tilde{\psi}(x) > t\}$. Choose now a coordinate system with origin at x_0 such that the N-th coordinate direction is given by $\nu(x_0)$. For $\delta > 0$ let

$$Q_\delta := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : |x_i| < r, i = 1, \dots, N\} \tag{2.9}$$

and $R_\delta := \{x \in Q_\delta : x_N = 0\} \subset \mathbb{R}^{N-1}$. Since Ω is a domain of class C^2 there exists $\delta > 0$ and a function $u \in C^2(R_\delta)$ such that

$$\Omega \cap Q_\delta = \{(x', x_N) \in R_\delta \times (-\delta, \delta) : x_N < u(x')\}. \tag{2.10}$$

Now we look at $\tilde{U}_t$. Note that the outer unit normal to $\tilde{U}_t$ at x_0 points in the direction of $-\operatorname{grad}\psi(x_0)$. By (2.8) $\operatorname{grad}\tilde{\psi}(x_0)$ has a non-zero component into the direction of $\nu(x_0)$. Hence by the implicit function theorem there exists $\eta > 0$ and a function $v \in C^1(R_\eta)$ such that

$$\tilde{U}_t \cap Q_\eta = \{(x', x_N) \in R_\eta \times (-\eta, \eta) : x_N < v(x')\}.$$

Setting $\varepsilon := \min\{\delta, \eta\}$ and $g := \min\{u, v\}$ we have

$$(\tilde{U}_t \cap \Omega) \cap Q_\varepsilon = \{(x', x_N) \in R_\varepsilon \times (-\varepsilon, \varepsilon) : x_N < g(x')\}.$$

Since the minimum of two C^1-functions is Lipschitz continuous it follows that g is Lipschitz continuous on R_ε. The situation is depicted in Figure 1. Hence $U_t = \tilde{U}_t \cap \Omega$ is Lipschitz near x_0, showing that U_t is Lipschitz.

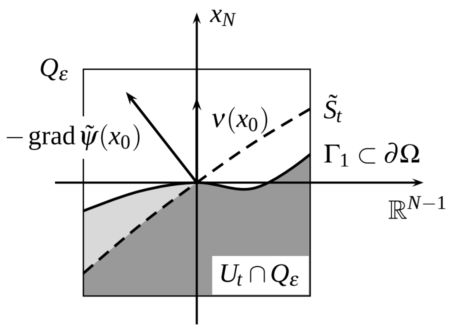

Fig. 1 Representation of U_t near a boundary point $x_0 \in \Gamma_1$.

(3) To prove the final assertion of the lemma let $K = \{x \in \overline{\Omega} : \psi(x) = m\}$. By the maximum principle $K \subset \partial\Omega$. By the Hopf boundary maximum principle (see [16, Theorem 2.15]) we know that $\partial u/\partial\nu \neq 0$ on K. Hence, as $\psi > 0$ in Ω

$$-\frac{\partial\psi}{\partial\nu}(z) = -\nu(z)\cdot\operatorname{grad}\psi(z) > 0$$

for all $z \in K$. By continuity of ν and $\operatorname{grad}\psi$ there exists $\alpha > 0$ such that

$$-\nu(z)\cdot\operatorname{grad}\psi(z) \geq \alpha > 0$$

for all $z \in K$. Since continuous functions are uniformly continuous on compact sets there exists $\delta_0 > 0$ such that

$$-\nu(z)\cdot\operatorname{grad}\tilde{\psi}(x) \geq \frac{\alpha}{2} > 0 \tag{2.11}$$

for all $z \in K$ and $x \in B(z,\delta_0)$. Similarly there exists $\delta_1 > 0$ such that $\nu(x)\cdot\nu(z) > 1/2$ for all $z \in K$ and $x \in B(z,\delta_1)\cap\partial\Omega$. Hence there exists $\delta > 0$ such that for each $z \in K$ there is an open cube $Q_{z,\delta}$ centred at z such that, if we choose a coordinate system with x_N in the $\nu(z)$ direction, then

(i) $Q_{z,\delta}$ has the form (2.9);
(ii) there exists $u \in C^2(R_{z,\delta})$ such that (2.10) holds;

Here $R_{z,\delta}$ is as before. We also use that by the implicit function theorem, $\partial\Omega$ is a graph over $\mathbb{R}^{N-1}$ as long as $\nu(x)\cdot\nu(z) \neq 0$. Then $Q_{z,\delta}$, $z \in K$, forms an open cover of the compact set K and therefore has a finite sub-cover. Hence there exist $z_1,\dots,z_n \in K$ such that $Q_{z_k,\delta}$ is an open cover of K. Since ψ attains a strict minimum on K there exists $t_1 \in (m,1)$ such that

$$S_t \subset V := \bigcup_{k=1}^{n} Q_{z_k,\delta}$$

for all $t \in (m,t_1]$. We now fix $t \in (0,t_1]$ and focus our attention on one particular cube, say $Q_\delta := Q_{z_k,\delta}$. As done above we choose a coordinate system with origin at $z := z_k$ and x_N-coordinate into the direction of $\nu(z)$. Then (2.11) reads

$$-\frac{\partial\tilde{\psi}}{\partial x_N}(x) \geq \frac{\alpha}{2} > 0 \tag{2.12}$$

for all $x \in Q_\delta$. If $S_t \cap Q_\delta \neq \emptyset$, then by the implicit function theorem there exists a subset $D \subset R_\delta$ and function $v \in C^1(D,\mathbb{R})$ such that $S_t \cap Q_\delta$ is the graph of v. The surface area of that graph is given by

$$\sigma(S_t) = \int_D \sqrt{1+|\operatorname{grad} v(x')|^2}\,dx'.$$

Since S_t is a level surface of $\tilde{\psi}$, the normal is given by

$$(-\operatorname{grad} v(x'),1) = \frac{1}{\frac{\partial\tilde{\psi}}{\partial x_N}(x',v(x'))}\operatorname{grad}\tilde{\psi}(x',v(x')),$$

so taking into account (2.12) we have

$$\sigma(S_t) = \int_{R_\delta} \frac{1}{|\frac{\partial\tilde{\psi}}{\partial x_N}(x',v(x'))|}|\operatorname{grad}\tilde{\psi}(x',v(x'))|\,dx' \leq \frac{2}{\alpha}\|\operatorname{grad}\tilde{\psi}\|_\infty\,\sigma(R_\delta)$$

We have also chosen δ such that $\partial\Omega\cap Q_\delta$ is the graph of a C^2-function u. Hence $\sigma(R_\delta) \leq \sigma(\partial\Omega\cap Q_\delta) \leq \sigma(\partial\Omega)$, and with the above

$$\sigma(S_t\cap Q_\delta) \leq \frac{2}{\alpha}\|\operatorname{grad}\tilde{\psi}\|_\infty\,\sigma(\partial\Omega)$$

Hence adding up over all n cubes we get

$$\sigma(S_t) \leq \frac{2n}{\alpha}\|\operatorname{grad}\tilde{\psi}\|_\infty\,\sigma(\partial\Omega),$$

completing the proof of the lemma. □

The assertions of the above lemma remain true under alternative assumptions. We discuss some possibilities below.

Remark 2.4 (a) For (1) and (3) of Lemma 2.3 above we only need that $\beta(x) \geq 0$ for all $x \in \Gamma_1$, not positivity everywhere. If we have pure Dirichlet boundary conditions, that is, $\Gamma_0 = \partial\Omega$, then (2) simply follows from Sard's theorem since $\overline{U}_t \subset \Omega$ for all $t \in (0,1)$.

(b) To show that U_t is Lipschitz in part (2) we used that grad $\tilde{\psi} \neq 0$ on Γ_1. This is guaranteed by the boundary conditions if $\beta(x) > 0$ for all $x \in \Gamma_1$. If $\beta = 0$ on part of Γ_1, then the argument does not work any more. We could try and overcome this by using Sard's theorem, and applying it to the extension $\tilde{\psi}$ of ψ. The idea is that almost all $t \in (m,1)$ are regular values of $\tilde{\psi}$. Recall that t is a regular value of $\tilde{\psi}$ if grad $\psi(x) \neq 0$ for all $x \in \tilde{S}_t$. In that case $\tilde{S}_t$ is a hyper-surface of class C^1, and by the boundary conditions

$$-\frac{\partial\psi}{\partial\nu}(z) = -\nu(z)\cdot\operatorname{grad}\psi(z) \geq 0$$

for $z \in \Gamma_1 \cap \tilde{S}_t$. If we choose a regular value t and $x_0 \in \tilde{S}_t \cap \Gamma_1$ then by the implicit function theorem we can find local representations of $\tilde{S}_t$ and Γ_1 as in the proof of (2) if we choose the N-th coordinate in the direction of

$$\nu(x_0) + \frac{\operatorname{grad}\tilde{\psi}(x_0)}{|\operatorname{grad}\tilde{\psi}(x_0)|}.$$

The problem is that we only know that $\tilde{\psi} \in C^1(\mathbb{R}^N)$ and thus we cannot apply Sard's theorem as attempted above, not even in dimension $N = 2$ (see [36]). However, if ψ has an extension $\tilde{\psi} \in C^N(\mathbb{R}^N)$, then Sard's Lemma (see [20, Theorem 3.1.3]) applies and the above arguments work. To ensure that the above is the case it is sufficient to assume that Ω is of class $C^{N,\alpha}$ and $\beta \in C^{N-1,\alpha}(\Gamma_1)$ for some $\alpha \in (0,1)$. Then $\psi \in C^{N,\alpha}(\overline{\Omega})$ has an extension $\tilde{\psi} \in C^{N,\alpha}(\mathbb{R}^N)$ as needed. Of course the above works if Ω and β are of class C^∞.

(c) If $N = 2$, then by the analyticity of the eigenfunction ψ and the boundary conditions grad $\psi(x) = 0$ for at most finitely many points in Ω. Hence by the implicit function theorem U_t is Lipschitz and (2.6) and (2.7) hold for all $t \in (m,1)$ except possibly a finite number of t.

3 A Lower Estimate for the First Eigenvalue

The purpose of this section is to give a lower estimate of the first eigenvalue of (2.1) in terms of the functional H_Ω introduced in the previous section. That estimate is the key to prove the main result of this paper. The proofs are slightly different for pure Robin boundary conditions. Since we are mainly concerned with such problems we start this section by looking at pure Robin conditions. At the end of the section we prove a similar estimate for problems involving Dirichlet boundary conditions (Theorem 3.5). We conclude the section by discussing some variational characterisations of $\lambda_1(\Omega)$.

We make the same assumptions on Ω and β as in Section 2, but assume that $\Gamma_1 = \partial\Omega$. In particular we assume that $\beta \in C^1(\partial\Omega)$ with $\beta(x) > 0$ for all $x \in \partial\Omega$. According to Remark 2.4 we could just require that $\beta(x) \geq 0$ on $\partial\Omega$ with $\beta(x) > 0$ somewhere if β and the domain are sufficiently smooth. We consider the convex subset $M_\beta := M_\beta(\Omega)$ of $C(\Omega)$ given by

$$M_\beta := \left\{u \in C(\Omega)\colon \limsup_{x\to z} \varphi(x) \leq \beta(z) \text{ for all } z \in \partial\Omega\right\},$$

(Functions in M_β are called *admissible repartitions* in [5, 6].) As in the previous section we normalise the first eigenfunction ψ of (1.1) such that it is positive with $\|\psi\|_\infty = 1$ and $m = \min\psi$. Finally let U_t and S_t be the level sets of ψ as defined in (2.4) and (2.5).

Theorem 3.1 *Let Ω be a bounded domain of class C^2 and $\lambda_1(\Omega)$ the first eigenvalue of (1.1). Then, with the above notation, for every $\varphi \in M_\beta$ there exists $t \in (m,1)$ such that $H_\Omega(U_t,\varphi) \leq \lambda_1(\Omega)$. Moreover,*

$$\lambda_1(\Omega) \geq \inf_{t\in(m,1)} H_\Omega(U_t,\varphi) \geq \inf_{U\,\subset\,\Omega \text{ open}} H_\Omega(U,\varphi) \tag{3.1}$$

for all $\varphi \in M_\beta$.

It is evident that (3.1) follows from the first assertion of the theorem. Hence, the remainder of this section is devoted to the proof of the first assertion. A very special role will be played by the function $|\operatorname{grad}\psi|/\psi$.

Remark 3.2 Note that $|\operatorname{grad}\psi|/\psi \in M_\beta$ only under very special conditions. We show that $|\operatorname{grad}\psi|/\psi \in M_\beta$ if and only if ψ is locally constant on $\partial\Omega$. Indeed, if ψ is locally constant on $\partial\Omega$, then $\partial\psi/\partial\nu = -|\operatorname{grad}\psi|$. By the boundary condition in (1.1) we have $|\operatorname{grad}\psi|/\psi \in M_\beta$. If $|\operatorname{grad}\psi|/\psi \in M_\beta$, then by the boundary condition and the continuity of $\operatorname{grad}\psi$

$$\frac{|\operatorname{grad}\psi|}{\psi} \leq \beta = -\frac{1}{\psi}\frac{\partial\psi}{\partial\nu} \leq \frac{|\operatorname{grad}\psi|}{\psi}$$

for all $x \in \partial\Omega$. Hence $\partial\psi/\partial\nu = -|\operatorname{grad}\psi|$ for all $x \in \partial\Omega$, implying that ψ is locally constant on $\partial\Omega$.

Given $\varphi \in C(\Omega)$ we define w as in Theorem 2.2, namely

$$w := \varphi - \frac{|\operatorname{grad}\psi|}{\psi}. \tag{3.2}$$

For the proof of Theorem 3.1 we need the differentiability of a function appearing in the representation (2.7) of $H_\Omega(U_t,\varphi)$. The statement remains valid under the assumptions of Section 2.

Lemma 3.3 *Suppose that the assumptions of Section 2 hold and that $\varphi\in C(\Omega)$ is non-negative such that $\varphi\in L_1(U)$ for every open set $U\subset\Omega$ with $\overline{U}\subset\Omega\cup\Gamma_1$. Let w be as in (3.2) and define*

$$F(t):=\int_t^1\frac{1}{\tau}\int_{S_\tau}w\,d\sigma\,d\tau$$

for all $t\in(m,1)$. Then F is absolutely continuous on $[\varepsilon,1)$ for all $\varepsilon\in(0,1)$ and

$$\frac{d}{dt}F(t)=-\frac{1}{t}\int_{S_t}w\,d\sigma$$

for almost all $t\in(0,1)$.

Proof Fix $\varepsilon\in(0,1)$. By assumption $\varphi\in C(\Omega)\cap L_1(U_\varepsilon)$. Applying the coarea formula (see [13, Section 3.4.3] or [24, Section 1.2.4])

$$\int_\varepsilon^1\frac{1}{\tau}\int_{S_\tau}\varphi\,d\sigma\,d\tau=\int_{U_\varepsilon}\frac{\varphi}{\psi}|\operatorname{grad}\psi|\,dx<\infty$$

and

$$\int_\varepsilon^1\frac{1}{\tau}\int_{S_\tau}\frac{|\operatorname{grad}\psi|}{\psi}\,d\sigma\,d\tau=\int_{U_\varepsilon}\frac{|\operatorname{grad}\psi|^2}{\psi^2}\,dx<\infty,$$

where we used that $|\operatorname{grad}\psi|/\psi$ is bounded on U_ε. Since the functions involved are all non-negative it follows that

$$f(\tau):=\frac{1}{\tau}\int_{S_\tau}w\,d\sigma=\frac{1}{\tau}\int_{S_\tau}\varphi\,d\sigma-\frac{1}{\tau}\int_{S_\tau}\frac{|\operatorname{grad}\psi|}{\psi}\,d\sigma$$

defines a function in $L_1((0,1))$. Hence

$$F(t)=\int_t^1 f(\tau)\,d\tau$$

is absolutely continuous on $[\varepsilon,1)$ and thus differentiable almost everywhere (see [31, Theorem 8.17]). Moreover,

$$F'(t)=-f(t)=-\frac{1}{t}\int_{S_t}w\,d\sigma$$

for almost all $t\in(\varepsilon,1)$. Since $\varepsilon\in(0,1)$ was arbitrary this completes the proof of the lemma. □

The last ingredient for the proof of Theorem 3.1 is the following estimate for w near $\partial\Omega$. It is used in a contradiction argument, whose details are completely omitted in [5–7]. We again look at the case where $\Gamma_1=\partial\Omega$.

Lemma 3.4 *Let $\varphi \in M_\beta$ and let w be as in (3.2). Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x) \leq \varepsilon$ for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta$.*

Proof Assume that $\varphi \in M_\beta$ and fix $\varepsilon > 0$. Since $|\operatorname{grad}\psi|/\psi$ is continuous and therefore uniformly continuous on the compact set $\overline{\Omega}$ there exists $\delta_0 > 0$ such that

$$\left|\frac{|\operatorname{grad}\psi(x)|}{\psi(x)} - \frac{|\operatorname{grad}\psi(z)|}{\psi(z)}\right| < \frac{\varepsilon}{2} \tag{3.3}$$

for all $x, z \in \overline{\Omega}$ with $|x - z| < \delta_0$. Fix $z \in \partial\Omega$. By assumption $\limsup_{x\to z}\varphi(x) \leq \beta(z)$. Hence there exists $r_z > 0$ such that $\sup_{x\in B(z,r_z)\cap\Omega}\varphi(x) \leq \beta(z) + \varepsilon/2$ and so

$$\varphi(x) - \beta(z) \leq \frac{\varepsilon}{2}. \tag{3.4}$$

for all $x \in B(z, r_z) \cap \Omega$. The balls $B(z, r_z)$, $z \in \partial\Omega$ form an open cover of $\partial\Omega$. By the compactness of $\partial\Omega$ we can select a finite sub-cover $B(z_i, r_i)$, $i = 1, \dots, n$, where $r_i := r_{z_i}$. We then choose $\delta \leq \min\{r_1, \dots, r_n, \delta_0\}$ such that $x \in \bigcup_{i=1}^n B(z_i, r_i)$ whenever $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta$. Fix now $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta$. By choice of δ there exists $i \in \{1, \dots, n\}$ with $x \in B(z_i, r_i)$. Since by the boundary conditions $\beta(z_i) \leq |\operatorname{grad}\psi(z_i)|/\psi(z_i)$ we get from (3.3) and (3.4)

$$\begin{aligned} w(x) = \varphi(x) - \frac{|\operatorname{grad}\psi(x)|}{\psi(x)} &\leq \varphi(x) - \beta_i(z_i) + \beta_i(z_i) - \frac{|\operatorname{grad}\psi(x)|}{\psi(x)} \\ &\leq \frac{\varepsilon}{2} + \frac{|\operatorname{grad}\psi(z_i)|}{\psi(z_i)} - \frac{|\operatorname{grad}\psi(x)|}{\psi(x)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since the above works for any choice of x under consideration the assertion of the lemma is proved. □

We are now ready to give a proof of Theorem 3.1.

Proof (Proof of Theorem 3.1) We give a proof by contradiction. Suppose that there exists $\varphi \in M_\beta$ such that

$$\lambda_1(\Omega) < H_\Omega(U_t, \varphi) \tag{3.5}$$

for all $t \in (m, 1)$. Set w as in (3.2) and F as in Lemma 3.3. Then by Theorem 2.2

$$\lambda_1(\Omega) < H_\Omega(U_t, \varphi) = \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\, d\sigma - 2F(t) - \int_{U_t} |w|^2\, dx\Big)$$

and therefore

$$2F(t) < \int_{S_t} w\, d\sigma - \int_{U_t} |w|^2\, dx \tag{3.6}$$

for almost all $t \in (m, 1)$. Using Lemma 3.3 we conclude that

$$\frac{d}{dt}\big(t^2 F(t)\big) = t\Big(-\int_{S_t} w\, d\sigma + 2F(t)\Big) < -t\int_{U_t} |w|^2\, dx \leq 0 \tag{3.7}$$

for almost all $t \in (m,1)$. Hence the function $t^2F(t)$ is strictly decreasing on $(m,1)$. Since $F(1) = 0$ and F is continuous there exist $\eta > 0$ and $t_0 \in (m,1)$ such that $F(t) > \eta$ for all $t \in (m,t_0]$. We now show that this contradicts the assumption that $\varphi \in M_\beta$. By Lemma 2.3 there exist $t_1 \in (m,t_0]$ and $c > 0$ such that $\sigma(S_t) \leq c\,\sigma(\partial\Omega)$ for all $t \in (m,t_1]$. If we set

$$\varepsilon := \frac{\eta}{c\,\sigma(\partial\Omega)},$$

then by Lemma 3.4 there exists $\delta > 0$ such that $w(x) \leq \varepsilon$ for all $x \in \Omega$ with $\operatorname{dist}(x,\partial\Omega) < \delta$. Since ψ attains a strict minimum on $\partial\Omega$ there exists $t \in (m,t_1]$ such that $\operatorname{dist}(x,\partial\Omega) < \delta$ for all $x \in S_t$. We can choose t such that (3.6) also holds. Hence, by (3.6) and our choice of ε, t and η

$$0 < 2\eta < 2F(t) < \int_{S_t} w\,d\sigma \leq \varepsilon\,\sigma(S_t) \leq \eta.$$

Since this is obviously a contradiction, (3.5) cannot be true for all $t \in (m,1)$. Hence there exists $t \in (m,1)$ such that $H_\Omega(U_t,\varphi) \geq \lambda_1(\Omega)$ as claimed. □

We next turn to problems involving Dirichlet boundary conditions, that is, problems of the form (2.1) with $\Gamma_0 \neq \emptyset$. The only difference in the proof is that it requires another contradiction argument. The main ideas are from [5, Section 8.3], where a proof in case $N = 2$ and an outline for $N \geq 3$ for pure Dirichlet boundary conditions was given. The set M_β used above will be replaced by the set of non-negative functions in $C(\Omega)$.

Theorem 3.5 *Let Ω be a bounded domain of class C^2 and $\lambda_1(\Omega)$ the first eigenvalue of (2.1) with $\Gamma_0 \neq \emptyset$. Then, with the above notation, for every $\varphi \in C(\Omega)$ non-negative there exists $t \in (0,1)$ such that $H_\Omega(U_t,\varphi) \leq \lambda_1(\Omega)$. Moreover,*

$$\lambda_1(\Omega) \geq \inf_{t\in(0,1)} H_\Omega(U_t,\varphi) \geq \inf_{\substack{U \subset \Omega \text{ open} \\ \overline{U}\cap\Gamma_0=\emptyset}} H_\Omega(U,\varphi) \tag{3.8}$$

for all $\varphi \in C(\Omega)$ non-negative.

Proof We give a proof by contradiction, assuming there exists $\varphi \in C(\Omega)$ non-negative such that (3.5) holds for all $t \in (0,1)$. Then, as in the proof of Theorem 3.1, we get (3.6) as well as (3.7). Hence, as before, the function $G(t) := t^2F(t)$ and $F(t)$ are positive and strictly decreasing on $(0,1)$. As a consequence

$$g(t) := \frac{1}{G(t)}$$

is strictly increasing on $(0,1)$. Since $F(t) > 0$ it follows from (3.6) that

$$\int_{S_t} w\,d\sigma > 2F(t) + \int_{U_t} |w|^2\,dx > 0$$

for almost all $t \in (0,1)$. By the Cauchy-Schwarz inequality and the coarea formula

$$\begin{aligned} G(t) = t^2 F(t) &= t\int_t^1 \frac{t}{\tau}\int_{S_\tau} w\,d\sigma\,d\tau < t\int_t^1\int_{S_\tau} w\,d\sigma\,d\tau \\ &= t\int_{U_t} w|\operatorname{grad}\psi|\,dx \le t\Big(\int_{U_t}|w|^2\,dx\Big)^{1/2}\Big(\int_{U_t}|\operatorname{grad}\psi|^2\,dx\Big)^{1/2} \end{aligned}$$

for all $t \in (0,1)$. Taking into account (3.7) we get

$$tg'(t) = -\frac{tG'(t)}{(G(t))^2} > \Big(\int_{U_t}|\operatorname{grad}\psi|^2\,dx\Big)^{-1}$$

for almost all $t \in (0,1)$. Fix now $t_1 \in (0,1)$. Since the last integral is a decreasing function of t we have

$$g'(t) > \frac{c}{t}$$

for almost all $t \in (0,t_1]$ if we set $c := \|\operatorname{grad}\psi\|_2^{-2}$. Since G is absolutely continuous and positive on $[\varepsilon,1)$ for all $\varepsilon \in (0,1)$, so is g. Using the fundamental theorem of calculus for such functions (see [31, Theorem 8.18])

$$g(t_1) \ge g(t_1) - g(\varepsilon) = \int_\varepsilon^{t_1} g'(\tau)\,d\tau > c\int_\varepsilon^{t_1}\frac{1}{\tau}\,d\tau = c(\log t_1 - \log\varepsilon)$$

for all $\varepsilon \in (0,t_1]$. Letting ε to zero we see that $-\log\varepsilon$ is bounded from above. As this is not true, the assertion of the theorem must be valid. □

As a corollary we get the following characterisation of $\lambda_1(\Omega)$. Special cases appear in [5].

Corollary 3.6 *Under the assumptions of Theorem 3.5*

$$\lambda_1(\Omega) = \max_{\substack{\varphi\in C(\Omega)\\ \varphi\ge 0}}\Big(\operatorname*{ess-inf}_{t\in(0,1)} H_\Omega(U_t,\varphi)\Big) = \max_{\substack{\varphi\in C(\Omega)\\ \varphi\ge 0}}\Big(\operatorname*{ess-inf}_{\substack{U\subset\Omega \text{ open}\\ \overline{U}\cap\Gamma_0=\emptyset}} H_\Omega(U,\varphi)\Big).$$

Proof The assertion follows from Theorem 3.5 if we set $\varphi := |\operatorname{grad}\psi|/\psi$ and then use Proposition 2.1. □

Remark 3.7 If $\lambda_1(\Omega)$ is the first eigenvalue of (1.1), for pure Robin problems we expect that, as above

$$\lambda_1(\Omega) = \mu(\Omega) := \sup_{\varphi\in M_\beta}\Big(\operatorname*{ess-inf}_{t\in(m,1)} H_\Omega(U_t,\varphi)\Big). \tag{3.9}$$

From Theorem 3.1 we have $\lambda_1(\Omega) \le \mu(\Omega)$, but we are not sure whether equality holds, unless $\varphi := |\operatorname{grad}\psi|/\psi \in M_\beta$ of course, in which case we have a maximum. From Remark 3.2 we know that $\varphi \in M_\beta$ if and only if ψ is locally constant on $\partial\Omega$. In the general case we need to interchange the order of supremum and infimum.

We can choose a sequence $\varphi_n \in C_c^\infty(\Omega)$ with $0 \leq \varphi_n \leq \varphi_{n+1} \leq \varphi$ for all $n \in \mathbb{N}$ and $\varphi_n \longrightarrow \varphi$ pointwise. Then by the monotone convergence theorem

$$\int_{S_t} \varphi_n \, d\sigma + \int_{U_t} |\varphi_n|^2 \, dx \to \int_{S_t} \varphi \, d\sigma + \int_{U_t} |\varphi|^2 \, dx$$

for all $t \in (m,1)$ as $n \to \infty$. Hence, from the definition (2.2) of H_Ω and Proposition 2.1 we see that $H_\Omega(U_t, \varphi_n) \to H_\Omega(U_t, \varphi) = \lambda_1(\Omega)$ for all $t \in (m,1)$. On the other hand Theorem 3.1 implies the existence of $t_n \in (m,1)$ such that $\lambda_1(\Omega) \geq H_\Omega(U_{t_n}, \varphi)$ for all $n \in \mathbb{N}$, but that does not necessarily imply (3.9).

4 Proof of the Main Result

In this section we give a proof of Theorem 1.1. The key is the estimate of $\lambda_1(\Omega)$ given in Theorem 3.1 applied to a function $\varphi \in M_\beta$ constructed as a rearrangement of an appropriate function on the ball. The same proof works for Dirichlet boundary conditions with obvious modifications using Theorem 3.5.

We start by looking at properties of (1.1) on a ball. We assume that $\beta \in (0, \infty)$ is a constant and that B is a ball of radius R. Without loss of generality we can assume that B is centred at the origin. We denote the first eigenvalue of (1.1) on B by λ_1^* and a corresponding eigenfunction by ψ^*. As in the previous section we normalise ψ^* such that $\psi^* > 0$ and $\|\psi^*\|_\infty = 1$. Since B is a ball the eigenfunction is radially symmetric, that is, $\psi^*(x) = v(|x|)$ for some function $v \in C^1([0,R])$. As in the previous section, the function

$$\varphi^* := \frac{|\operatorname{grad} \psi^*|}{\psi^*} \tag{4.1}$$

will play a crucial role. By the radial symmetry ψ^* is constant on ∂B and thus by Remark 3.2 $\psi^* \in M_\beta(B)$. The idea then is to construct a rearrangement φ of φ^* lying in $M_\beta(\Omega)$, and then to use Theorem 3.1 to prove Theorem 1.1. We next establish some properties of φ^*. Since φ^* is radially symmetric we only need to look at the radial function

$$g(|x|) := \varphi^*(x)$$

for $x \in B$.

Lemma 4.1 *The function $g\colon (0,R) \to (0,\infty)$ is strictly increasing.*

Proof Since ψ^* satisfies (1.1), the function v defined above is a positive solution of the radial equation

$$v''(r) + \frac{N-1}{r} v'(r) + \lambda_1^* v(r) = 0 \quad \text{for } r \in (0,R].$$

It is easily verified that

$$v(r) = cr^{-(N/2-1)} J_{N/2-1}(\sqrt{\lambda^*} r)$$

for all $r \in (0, R]$, where J_n is the Bessel function of index n and c a normalising constant. Since ψ^* is decreasing in the radial direction and by elementary properties of Bessel functions (see [35, page 45])

$$|\operatorname{grad} \psi^*(x)| = -v'(|x|) = c\sqrt{\lambda^*} |x|^{-(N/2-1)} J_{N/2}(\sqrt{\lambda^*}|x|)$$

for all $x \in B$. Hence

$$g(r) = \sqrt{\lambda^*} \frac{J_{N/2}(\sqrt{\lambda^*} r)}{J_{N/2-1}(\sqrt{\lambda^*} r)}$$

for all $r \in (0, R)$. If j_n, $n \in \mathbb{N}$, are the positive zeros of the Bessel function $J_{N/2-1}$, then it is known that

$$\frac{J_{N/2}(r)}{J_{N/2-1}(r)} = -\sum_{n=0}^{\infty} \Big(\frac{1}{r - j_n} + \frac{1}{r + j_n} \Big)$$

whenever $r \neq j_n$ for all $n \in \mathbb{N}$ (see [35, page 498]). Since each of the terms in the above series is a strictly decreasing function of r between the zeros of $J_{N/2-1}$, it follows that g is strictly increasing as claimed. $\square$

Assume now that Ω is a domain of class C^2 and that the ball B has the same volume as Ω. We next define $\varphi \in M_\beta(\Omega)$ by constructing a suitable rearrangement of φ^*. As in the previous section we let $\psi > 0$ be the eigenfunction to $\lambda_1(\Omega)$ with $\psi > 0$, $\|\psi\|_\infty = 1$ and $0 < m = \min_{x \in \overline{\Omega}} \psi(x)$. As before, set $U_t := \{x \in \Omega : \ \psi(x) > t\}$ and $S_t := \{x \in \Omega : \ \psi(x) = t\}$. In what follows we denote the ball of radius r centred at the origin by B_r. We let $r(t)$ be the radius of the ball with the same volume as U_t. Since Ω and B have the same volume and $U_m = \Omega$ we have $r(m) = R$. For $x \in S_t$ and $t \in (m, 1]$ we now define

$$\varphi(x) := g(r(t)).$$

Since Ω is a disjoint union of S_t, $t \in (m, 1]$, the function φ is well defined.

Lemma 4.2 *The function φ constructed above lies in $M_\beta(\Omega)$. Moreover,*

$$\lambda_1^* = H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) \tag{4.2}$$

for all $t \in (m, 1)$.

Proof Since g is increasing it follows that $\{x \in \Omega : \varphi(x) > t\} = \Omega \setminus \overline{U}_t$ and $\{x \in \Omega : \varphi(x) < t\} = U_t$ are open in Ω for every $t \in \mathbb{R}$. Hence φ is continuous on Ω. Moreover, by construction and since g is increasing with $g(R) = \beta$ we have $\varphi(x) \leq \beta$ for all $x \in \Omega$. Hence $\varphi \in M_\beta(\Omega)$ as claimed. We next prove (4.2). The first equation follows by (4.1) and Proposition 2.1. Since by construction the level sets of φ^* and φ have the same volume

$$\int_{U_t} |\varphi|^2 \, dx = \int_{B_{r(t)}} |\varphi^*|^2 \, dx \tag{4.3}$$

for all $t \in (m,1]$ (see [24, Section 1.2.3]). As $|B_{r(t)}| = |U_t|$, the isoperimetric inequality (see [4] or [15, Theorem 3.2.43]) implies that $\sigma(\partial B_{r(t)}) \leq \sigma(\partial U_t)$ for all $t \in (m,1]$. As $\varphi(x) = g(r(t)) \leq \beta$ for $x \in S_t$ we have

$$\begin{aligned} \int_{\partial B_{r(t)}} \varphi^* \, d\sigma &= g(r(t))\, \sigma(\partial B_{r(t)}) \leq g(r(t))\, \sigma(\partial U_t) \\ &= \int_{S_t} g(r(t))\, d\sigma + \int_{\partial_e U_t} g(r(t))\, d\sigma \leq \int_{S_t} \varphi \, d\sigma + \int_{\partial_e U_t} \beta \, d\sigma. \end{aligned} \tag{4.4}$$

Using the definition (2.2) of H_B and H_Ω inequality (4.2) follows. □

Remark 4.3 If U_t is a ball and $\sigma(\partial_e U_t) = 0$, then it is evident from (4.3) and (4.4) that there is equality in (4.2). From (4.4), the converse is also true, at least if U_t is sufficiently smooth since there is equality in the isoperimetric inequality if and only if U_t is a ball (see [8, Theorem 10.2.1]). Recall that U_t is Lipschitz for almost every $t \in (m,1)$ by Lemma 2.3. From the above it should be possible to prove a uniqueness result for the minimising domain as mentioned at the end of the introduction.

Now we can give a proof of Theorem 1.1 under the assumption that Ω is a bounded domain of class C^2: If $\varphi \in M_\beta(\Omega)$ is the function constructed above, then by Theorem 3.1 there exists $t \in (m,1)$ such that $\lambda_1(\Omega) \geq H_\Omega(U_t, \varphi)$. But then (4.2) implies that

$$\lambda_1(\Omega) \geq H_\Omega(U_t, \varphi) \geq H_B(B_{r(t)}, \varphi^*) = \lambda_1^*,$$

completing the proof of Theorem 1.1 for bounded domains of class C^2.

For the final step in the proof of Theorem 1.1 assume now that Ω is a bounded Lipschitz domain. This means that $\partial\Omega$ is locally the graph of a Lipschitz function as explained in the proof of Lemma 2.3. It is shown in [25] or [12, Theorem 5.1] that Ω can be approximated from the outside by a sequence of domains Ω_n such that $|\Omega_n| \to |\Omega|$ and the same charts representing $\partial\Omega$ also work for $\partial\Omega_n$. On top of that we can choose Ω_n such that the boundary measure locally converges. More precisely, there exists a cover of cubes Q_i, $i = 1, \dots, m$, of the form (2.9) such that $\partial\Omega_n, \partial\Omega \subset \bigcup_{i=1}^m Q_i$. Moreover, there exist Lipschitz functions $u_i : R_i \to \mathbb{R}$ as in

(2.10). Finally there exist functions $u_{i,n} \in C^\infty(R_i)$ such that a property similar to (2.10) holds with $u_{i,n} \to u_i$ uniformly, $\|\operatorname{grad} u_{i,n}\|_\infty \leq \|\operatorname{grad} u_i\|_\infty$ for all $n \in \mathbb{N}$ and $\operatorname{grad} u_{i,n} \to \operatorname{grad} u_i$ almost everywhere. Note that Lipschitz functions are differentiable almost everywhere by Rademacher's theorem (see [13, Section 3.1.2]), so the above statements on the gradients make sense. In particular we get

$$\sqrt{1+|\operatorname{grad} u_{i,n}|^2} \to \sqrt{1+|\operatorname{grad} u_i|^2} \tag{4.5}$$

almost everywhere for all $i = 1,\dots,m$. Now it follows from [9, Theorem 4.4 and 6.2] that $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ (because of the boundedness of $\|\operatorname{grad} u_{i,n}\|_\infty$ and (4.5) we have $g \equiv 1$ in our situation). Denote by B_n the ball of the same volume as Ω_n centred at the origin. Since $|\Omega_n| \to |\Omega|$ we conclude from the above results in [9] also that $\lambda_1(B_n) \to \lambda_1(B)$. As Ω_n is of class C^∞ we have

$$\lambda_1(B_n) \leq \lambda_1(\Omega_n)$$

for all $n \in \mathbb{N}$ by what we have proved already. Passing to the limit the assertion of Theorem 1.1 follows for Lipschitz domains.

If we deal with Dirichlet boundary conditions we have to replace the above approximation argument. We assume that Ω is a domain of finite volume (not necessarily bounded). Then there exists a sequence (Ω_n) of domains of class C^∞ such that $\overline{\Omega}_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$. Let B_n be a ball of the same volume as Ω_n. Then, by the above result $\lambda_1(B_n) \leq \lambda_1(\Omega_n)$ for all $n \in \mathbb{N}$. Then $\lambda_1(B_n) \to \lambda_1(B)$ and $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ (see for instance [11, Section 4]). Hence $\lambda_1(B) \leq \lambda_1(\Omega)$ if Ω is an arbitrary domain of finite measure and B a ball with the same measure.

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