What is modular rep theory?

Talk structure.

§1. Introduction and motivation.

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§1. Introduction and motivation

- Modular rep. theory = rep. theory over a field k of prime characteristic p.

- Hugely different (often more difficult) than over fields of characteristic 0 (e.g. $\mathbb{C}$):

(1) Semisimplicity fails: Maschke's theorem for reps of finite groups G fails (proof involves dividing by $|G|$).

e.g. $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle \curvearrowright V = kx \oplus ky$

by $g \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then

V is reducible but indecomposable.

Such reps abound for more general G (e.g. algebraic groups).

(2) New symmetry: k has funny arithmetic, esp. the "freshman's dream",

$$(a+b)^p = a^p + b^p \text{ for } a, b \in k.$$

This often yields submodules which don't exist in characteristic 0 (examples later).

Underlying mechanism: Frobenius endomorphism

$$Fr: G \longrightarrow G$$

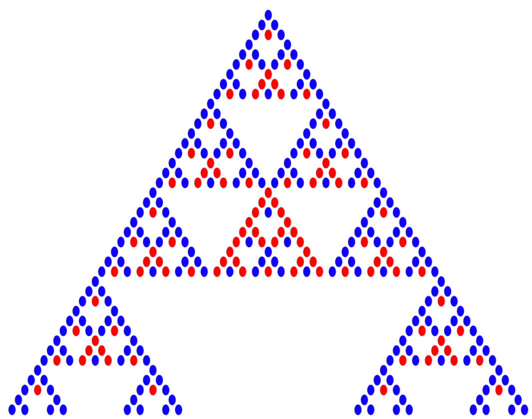
and associated Frobenius twist functor on Rep G $\rightsquigarrow$ "Fractal" structure.

(3) Subtler geometric connections: finding character formulas for simple modules has been a guiding problem in rep theory.

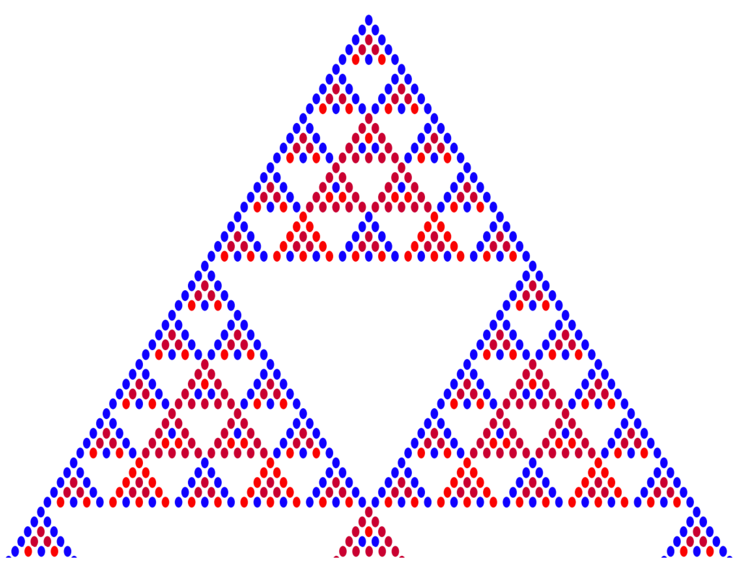
<u>Characteristic zero</u>: deep geometric proofs of Kazhdan–Lusztig conjecture by Brylinski–Kashiwara and Beilinson–Bernstein (1980's)

<u>Characteristic p</u>: problem still in the process of being solved! (Lusztig's conjecture has led the way.)

- Goal today: to explore some of these features, emphasising $G = SL_2$.
- Along the way, we will encounter + explain the following pictures:



$p = 3$



$p = 5$

- These images are from:

Williamson, G. (2020). Modular representations and reflection subgroups. *arXiv preprint arXiv:2001.04569*.

We follow exposition there and list other references at the end.

§2. Foundations

- We fix $k = \overline{k}$ an algebraically closed field of characteristic $p > 0$. (For conceteness can take $k = \overline{\mathbb{F}_p}$.)
- For us, an <u>algebraic group</u>/k is a group G with the structure of an affine k-variety,

$$\underbrace{\begin{array}{l} m: G \times G \longrightarrow G, \ (g,h) \longmapsto gh \\ \iota: G \to G, \ g \mapsto g^{-1} \end{array}}_{\text{regular maps}}$$

- A <u>homomorphism</u> of algebraic groups $G \to H$ should respect both structures: a regular group homomorphism.

- <u>Examples:</u> $\mathbb{G}_a = (k, +) = \mathbb{A}^1_k$,

$$\mathbb{G}_m = (k^\times, \times) = \mathbb{A}^1_k - \{0\},$$

(and more generally $\mathbb{G}_m^r$ = torus, $r \geq 1$)

$$GL_n(k) = D(\det) \underset{\text{open}}{\subseteq} \mathbb{A}^{n^2}_k.$$

- For any scheme X defined over $\mathbb{F}_p$, have a Frobenius endomorphism,

$$Fr : X \longrightarrow X.$$

In the case of affine varieties, given by p-th power map on coordinates,

e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$

on $Mat_{2\times 2}(k) = \mathbb{A}^4_k$.

- <u>Special case</u>: alg. group hom. $Fr: G \to G$.

- An <u>algebraic representation</u> of G is an alg. group hom
$$\varphi: G \longrightarrow GL(V) \cong GL_n(k)$$
for some finite-dim k-vector space V. This amounts to a group hom.
$$\varphi(g) = \left(z_{ij}(g)\right)_{i,j=1,\dots,n}$$
such that the $z_{ij}: G \longrightarrow k$ are regular functions.

§3 Examples for SL_2, Chevalley's Thm

- Let $G = SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_{2\times 2}(k) : ad-bc = 1 \right\}$.

- Examples of algebraic representations:

  (i) $SL_2 \longrightarrow GL_1(k)$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto 1$, the trivial rep.

(ii) $SL_2 \hookrightarrow GL_2(k)$, natural rep.

(iii) $SL_2 \longrightarrow GL_3(k)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

- In fact, both (i) and (iii) can be cooked up from (ii): let

$$V = kX \oplus kY = \text{nat}.$$

Then

$$\Lambda^2(V) = k(X \wedge Y) = \det$$

has $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (X \wedge Y) = (aX + cY) \wedge (bX + dY)$

$$= (ad - bc) X \wedge Y = X \wedge Y,$$

so it's trivial and we recover (i).

But also have

$$S^2(V) = kX^2 \oplus kXY \oplus kY^2$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X^2 = (aX + cY)^2$$
$$= a^2X + 2ac\,XY + c^2Y^2,$$

$$\vdots$$

Calculations show that we recover (iii).

- More generally, we have

$$S^n(V) = \nabla_n = kX^n \oplus kX^{n-1}Y \oplus \cdots \oplus kY^n$$

of dimension $n+1$, for all $n \geq 0$.

Then (i), (ii), (iii) are precisely ∇_0, ∇_1, ∇_2, respectively.

- Let $M = \mathfrak{sl}_2$-module.

  By restriction, M is a module for the maximal torus

$$T = \left\{ \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} : z \in k^\times \right\}$$

$$\rightsquigarrow M = \bigoplus_{n \in \mathbb{Z}} M_n \text{ where}$$

weight

$$M_n = \left\{ m \in M : \begin{pmatrix} z & \\ & z^{-1} \end{pmatrix} m = z^n m \ \ \forall z \in k^\times \right\}$$

weight space.

- <u>Key fact:</u> ∇_n has a unique simple submodule

$$L_n = \operatorname{soc} \nabla_n \xhookrightarrow{i} \nabla_n,$$

of highest weight n, where $\operatorname{coker}(i)$ has a comp. series by L_m's for $m < n$.

- If $n! \neq 0$ in k, then $\nabla_n \cong V_n^{\cdot\cdot}$ and we deduce $\nabla_n = L_n$ is simple. In fact, in characteristic zero,

$$\{\text{simple } SL_2\text{-modules}\}/\cong \;\overset{(*)}{\longleftrightarrow}\; \{\nabla_n\}_{n\geq 0}.$$

- But in characteristic p, story not so simple (pun intended!)
- <u>Example:</u> $k = \overline{\mathbb{F}_3}$. The formulas

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X^3 = (aX+cY)^3 = a^3X^3 + c^3Y^3,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y^3 = (bX+dY)^3 = b^3X^3 + d^3Y^3$$

Show that $kX^3 \oplus kY^3$ is a proper submodule of $\nabla_3 = kX^3 \oplus kX^2Y \oplus kXY^2 \oplus kY^3$. In fact this is simple, $L_3 = kX^3 \oplus kY^3$.

- How did L_3 arise here? Given an SL_2-rep

$$SL_2 \xrightarrow{\varphi} GL(M),$$

consider the rep afforded by precomposing w/ Frobenius:

$$SL_2 \xrightarrow{Fr} SL_2 \xrightarrow{\varphi} GL(M).$$

We obtain the <u>Frobenius twist</u> $M^{(1)}$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting as $Fr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}$.

- <u>Check</u>: $L_3 \cong \nabla_1^{(1)} = L_1^{(1)}$ for $p=3$.
- Now the correct reformulation of (*) for k of any characteristic:

$$\{\text{simple } SL_2\text{-modules}\}/\cong \;\longleftrightarrow\; \{L_n\}_{n\geq 0}.$$

- <u>General theorem [Chevalley]</u>: Let $T \subseteq G$ be a maximal torus (product of copies of $\mathbb{G}_m$), $X = \mathrm{Hom}(T, \mathbb{G}_m)$. Then isoclasses of simple G-modules are classified by an explicit set $X_+ \subseteq X$ of <u>dominant</u> weights.

- Caution: While the parameter set X_+ does not vary with p, the structure of the L_λ, $\lambda \in X_+$, certainly does (as we have seen).

§4 Characters and Pascal's Δ

- Generalising what we saw above: if $T \subseteq G$ is a maximal torus, then any G-module M admits a decomp.

$$M = \bigoplus_{\lambda \in X} M_\lambda \quad \text{λ-weight space.}$$

where $M_\lambda = \{ m \in M : tm = \lambda(t)m \ \forall t \in T \}$.

- The character of M is

$$\mathrm{Ch}\, M = \sum_{\lambda \in X} (\dim M_\lambda) e^\lambda \in \mathbb{Z}[X].$$

- After dimension, perhaps the most basic attribute of M.

- Ch is additive on exact sequences and satisfies

$$\mathrm{ch}(M \otimes N) = (\mathrm{ch}\, M)(\mathrm{ch}\, N)$$

if we define $e^{\lambda} \cdot e^{\mu} = e^{\lambda+\mu}$.

- Example: Recall $\nabla_n = kX^n \oplus kX^{n-1}Y \oplus \cdots \oplus kY^n$. Then

$$\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \cdot X^i Y^{n-i} = (zX)^i (z^{-1}Y)^{n-i}$$

$$= z^{2i-n} X^i Y^{n-i},$$

so $X^i Y^{n-i} \in (\nabla_n)_{2i-n}$ and hence see that ∇_n has 1-dimensional non-zero weight spaces for

$$-n, -n+2, \ldots, n-2, n \in \mathbb{Z},$$

i.e. $\mathrm{ch}\, \nabla_n = e^{-n} + e^{-n+2} + \cdots + e^{n-2} + e^{n}$

$$= \frac{e^{n} - e^{-n-2}}{1 - e^{-2}}.$$

- There exist analogues of the ∇_n for all reductive groups, and ch ∇_n is given in general by Weyl's character formula.
- Assume $p=3$. The following depicts the submodules $L_n \subseteq \nabla_n = kX^n \oplus \cdots \oplus kY^n$, for $0 \le n \le 6$:

$$k$$

$$kX \qquad kY$$

$$kX^2 \qquad kXY \qquad kY^2$$

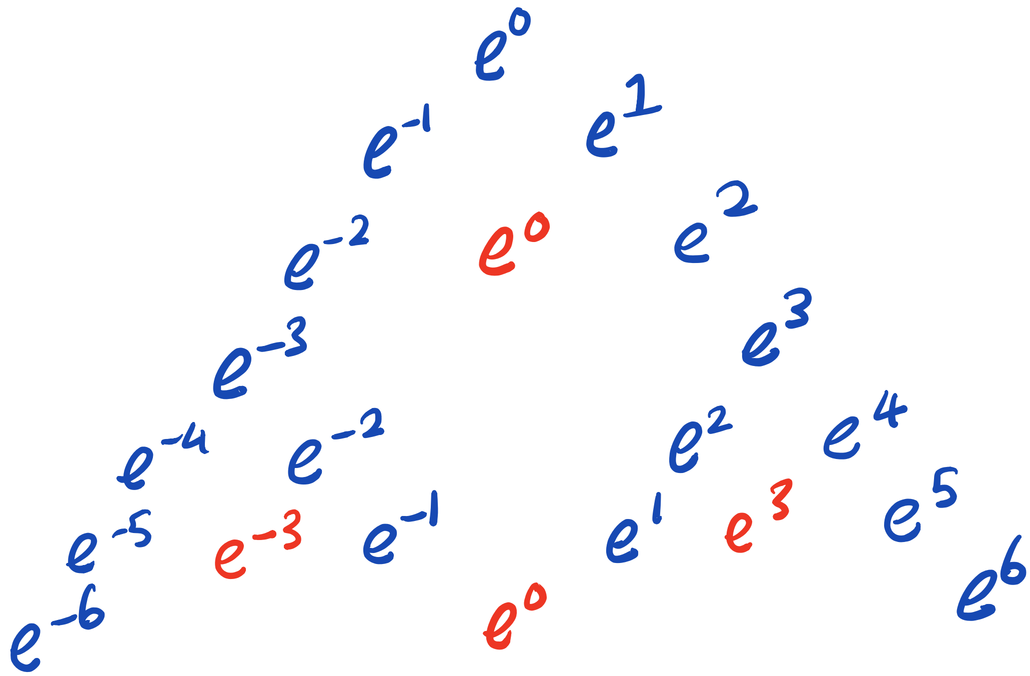
$$kX^3 \qquad\qquad kY^3$$

$$kX^4 \quad kX^3Y \qquad kY^3X \quad kY^4$$

$$kX^5 \quad kX^4Y \quad kX^3Y^2 \qquad kX^2Y^3 \quad kXY^4 \quad kY^5$$

$$kX^6 \qquad\qquad kX^3Y^3 \qquad\qquad kY^6$$

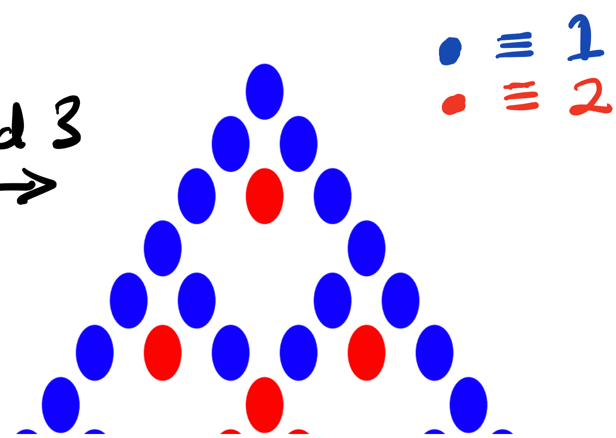
- From this we can read characters off directly:



This should recall our previous picture:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$\xrightarrow{\text{mod } 3}$



Let us note for now that this diagram is obtained by reducing Pascal's triangle (modulo 3) and coding congruence classes 0, 1, 2 as white, blue, red (resp.)

- How does modular Pascal's Δ connect to characters?

§5 Frobenius kernels, Steinberg $\otimes$-theorem

- Motivation: Suppose $N \trianglelefteq H$ are finite groups, and let us assume:

  All simple N-modules extend to H-modules $(\dagger)$

- Clifford: If V is a simple H-module, then $V|_N$ is semisimple with H-conjugate simple N-summands, all isomorphic (by $(\dagger)$):

$$V|_N \cong V' \oplus \cdots \oplus V'.$$

- Then

$$\mathrm{Hom}_N(V', V) \otimes V' \xrightarrow{\cong} V,$$
$$f \otimes v' \longmapsto f(v')$$

is an iso. of H-modules (clearly surjective, then compare dimensions).

- **Upshot:** Simple H-module $\cong$ (Simple H/N-mod) $\otimes$ (Simple N-mod)?

- **Back to algebraic groups:** Assume technical conditions on G: semisimple, simply connected.

- Can consider an exact sequence:

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{Fr} G \longrightarrow 1$$

Frobenius kernel = "N"

- **Curtis:** There is an explicit subset $X_1 \subseteq X_+$ such that

p-restricted weights

$$\{\text{simple } G_1\text{-modules}\}/\cong \;\longleftrightarrow\; \{L_\lambda|_{G_1}\}_{\lambda \in X_1}.$$

- **Theorem [analogue to upshot]:** If $\lambda = \mu + \nu \in X_1$ with $\mu \in X_1$, $\nu = p\nu' \in X_+$ then $L_\lambda \cong L_\mu \otimes L_{\nu'}^{(1)}$.

simple over $G \cong G/G_1 =$ "H/N"

simple over $G_1 =$ "N"

- In our setting, every $\lambda \in X_+$ can be written $\lambda = \lambda_0 + p\lambda_1 + \dots + p^r\lambda_r$, $\lambda_i \in X_1$.

- Corollary [Steinberg]:

$$L_\lambda \cong L_{\lambda_0} \otimes L_{\lambda_1}^{(1)} \otimes \cdots \otimes L_{\lambda_r}^{(r)}.$$

- Example: $G = SL_2$ has

$$X = \mathbb{Z} \supseteq X_+ = \mathbb{Z}_{\geq 0} \supseteq X_1 = \{0, 1, \ldots, p-1\}.$$

So for any $n \in \mathbb{Z}_{\geq 0}$ we take its p-adic expansion, $n = n_0 + n_1 p + \cdots + n_r p^r$, and write

$$L_n = L_{n_0} \otimes L_{n_1}^{(1)} \otimes \cdots \otimes L_{n_r}^{(r)}.$$

- Since $L_m = \nabla_m$ for $0 \leq m \leq p-1$, now have

$$\mathrm{ch}\, L_n = (\mathrm{ch}\, \nabla_{n_0})(\mathrm{ch}\, \nabla_{n_1})^{(1)} \cdots (\mathrm{ch}\, \nabla_{n_r})^{(r)}$$

$$(*) \quad = \prod_{i=0}^{r} \left(e^{-n_i} + e^{-n_i+2} + \cdots + e^{n_i-2} + e^{n_i}\right)^{(i)},$$

where $(e^m)^{(i)} = e^{p^i m}$ is extended linearly.

- We can now ask when $(L_n)_{n-2j} \neq 0$, i.e. when e^{n-2j} appears in $\operatorname{ch} L_n$.

- Write $j = j_0 + j_1 p + \cdots + j_r p^r$ in base p.

It is visible from the product (*) that to get e^{n-2j}, we need

$$j_i \leq n_i \quad \text{for all } 0 \leq i \leq r.$$

- Another way of phrasing this: no p-adic carries when adding j to $n-j$.

- <u>Kummer:</u> $v_p \binom{n}{j} = \#$ p-adic carries when adding j to $n-j$.

- So $(L_n)_{n-2j} \neq 0 \iff v_p \binom{n}{j} = 0$
$$\iff \binom{n}{j} \not\equiv 0 \pmod{p},$$
solving the modular Pascal mystery!

- Other explanations possible: Shapovalov form + Jantzen filtration.

(Additional) References

Books:

J. C. Jantzen. *Representations of Algebraic Groups*. American Mathematical Society, Providence, RI, 2003.

T. A. Springer. *Linear Algebraic Groups*, volume 9 of *Progress in mathematics*. Birkhaüser, 1981.

Online notes:

I. Losev. Lectures on Representation Theory. https://gauss.math.yale.edu/~il282/RT/

Articles:

J. Ciappara and G. Williamson. Lectures on the geometry and modular representation theory of algebraic groups. To appear in *Journal of the Australian Mathematical Society*, 2020.

G. Williamson. Algebraic representations and constructible sheaves. *Japanese Journal of Mathematics*, 12(2):211–259, 2017.