# Constructions with vectr bundles

## Talk plan

§1 Recap
§2 Associated fibre bundles
§3 Pullback bundles
§4 Clutching functions

## §1 Recap

- A rank $n$ <u>vector bundle</u> over $\mathbb{F}$ is a map

$$\underset{\text{total space}}{p: E} \longrightarrow \underset{\text{base space}}{B}$$

along with the structure of an $n$-dimensional $\mathbb{F}$-vector space on each fibre $p^{-1}(b)$, such that $p$ is <u>locally trivial</u>: $\exists$ open cover $\{U_i\}$ of $B$ with homeomorphisms in commutative squares

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\cong", "h_i"'] & U_i \times \mathbb{F}^n \\ p^{-1}(b) \arrow[u] \arrow[r, "\cong", "\text{linear}"'] & \{b\} \times \mathbb{F}^n \arrow[u] \end{tikzcd}$$

- A <u>section</u> of $p$ is a map $s: B \longrightarrow E$ with $ps = 1_B$. There is always a <u>zero section</u> $s_0$.

- Given $E_1 \to B$, $E_2 \to B$, have $E_1 \oplus E_2$, $E_1 \otimes E_2$ over $B$.

Construct using

$$\left\{\begin{matrix}\text{rank } n \\ \text{vector bundles} \\ \text{on } B\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{gluing functions} \\ g_{ij}: U_i \cap U_j \to GL_n(\mathbb{R})\end{matrix}\right\}$$

- An <u>inner product</u> on $E \to B$ is a map
$$\langle -,- \rangle : E \times E \longrightarrow \mathbb{R}$$
such that each restriction to a fibre is a real inner product
$$\langle -,- \rangle_b : p^{-1}(b) \times p^{-1}(b) \longrightarrow \mathbb{R}.$$
(If $\mathbb{F} = \mathbb{C}$ have Hermitian inner products.)

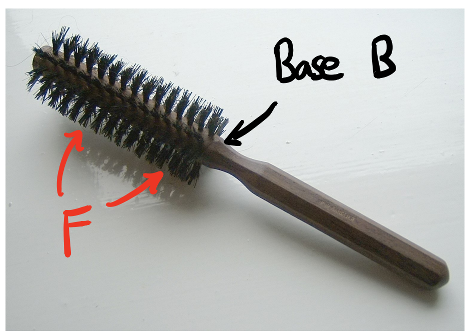
- <u>Prop:</u> Every vector bundle over a paracompact (e.g. compact Hausdorff or metric space) base admits an inner product.

- Suppose have
$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[dr, "p_0"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$
vector bundles

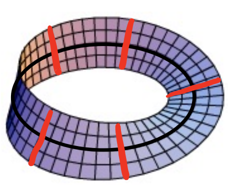
We say $E_0$ is a <u>vector subbundle</u> of $E$ in case $p_0^{-1}(b) \subseteq p^{-1}(b)$ is a vector subspace for all $b \in B$. Also then have a <u>quotient bundle</u>
$$E/E_0 \longrightarrow B.$$

## §2 Associated fibre bundles

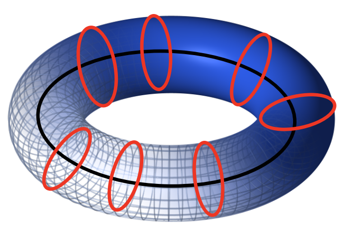
- Vector bundles have fibres of the form $\mathbb{F}^n$. Much more generally, fibre bundles have fibres of the form $F$, where $F$ is an arbitrary top. space.



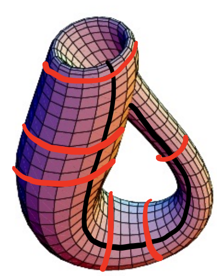

- Examples:




$B = S^1$
$F = I$




$B = S^1$
$F = S^1$




$B = S^1$
$F = S^1$

- There are various procedures for creating fibre bundles from vector bundles.

- <u>Examples:</u> Let $E \to B$ an $\mathbb{R}$-vector bundle.

(1) $E \to B$ afforded an inner product $\rightsquigarrow$ length function $\|\cdot\|: E \to \mathbb{R}_{\geq 0} \rightsquigarrow$

$$S(E) = \{e \in E : \|e\| = 1\} \longrightarrow B,$$
$$D(E) = \{e \in E : \|e\| \leq 1\} \longrightarrow B,$$

bundles of spheres and discs, respectively.

<u>Exercise:</u> identify $S(\text{canonical} \longrightarrow \mathbb{R}P^n)$.

(2) $P(E) = \{\text{space of lines in fibres of } E\}$

$$= S(E) / (v \sim -v \text{ in } p^{-1}(b) \text{ for all } v, b)$$
$$\downarrow \quad \underline{\text{projective bundle}}$$
$$B.$$

If $E \to B$ has rank $n$, then $P(E) \to B$ is a bundle with fibres $\mathbb{R}P^{n-1}$.

(3) For $k \leq n = \text{rank}(E)$, let $F_k(E) \subseteq P(E)^k$ be the subspace of $k$-tuples of mutually orthogonal lines in fibres of $E \to B$.

Thus the fibres of $F(E) = F_n(E)$ can be identified with

$$F(\mathbb{R}^n) = \text{(complete) flag manifold of } \mathbb{R}^n.$$

(4) There is a <u>Grassmann bundle</u> $G_k(E) \to B$ whose fibres are the sets of $k$-dimensional linear subspaces in fibres of $E \to B$.

(So $G_1(E) = P(E)$.)

## §3 Pullback bundles

- <u>Prop</u>: Suppose we have a setup

$$\begin{tikzcd} & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

vector bundle

map

Then the diagram can be completed,

bundle morphism

pullback vector bundle

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

such that $f'$ induces isomorphisms $(p')^{-1}(a) \cong p^{-1}(f(a))$ for all $a \in A$. $p'$ is unique with this property, up to iso.

<u>Proof:</u> Existence: let

$$\begin{tikzcd} E' = \{(a,e) \in A \times E : p(e) = f(a)\} \arrow[r, "f'"] \arrow[d, "p'"] & E \\ A & \end{tikzcd}$$

$= $ pullback of diagram in Top.

Then $(p')^{-1}(a) \xrightarrow{f'} \{e \in E : p(e) = f(a)\} = p^{-1}(f(a))$ is clearly an iso.

Let $\Gamma_f = $ graph of $f = \{(a, f(a))\} \subseteq A \times B$

$$\begin{tikzcd} \{(a, f(a))\} \arrow[d, "\cong"] \\ A \end{tikzcd}$$

and consider $E' \longrightarrow \Gamma_f$, $(a,e) \mapsto (a, p(e))$
$= (a, f(a))$.

Then $(E' \longrightarrow \Gamma_f)$ = res. of v.b.
$1 \times p : A \times E \longrightarrow A \times B$,
over $\Gamma_f$

so $p' = (E' \longrightarrow \Gamma_f \cong A)$ is a v.b.

Uniqueness: use universal property in Top.

- <u>Upshot</u>: A map $f: A \longrightarrow B$ yields

$$f^*: \mathrm{Vect}^n(B) \longrightarrow \mathrm{Vect}^n(A).$$

- <u>Basic properties</u>: (1) $(fg)^* = g^* f^*$,
(2) $\mathrm{id}^* = \mathrm{id}$,
(3) $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$,
(4) $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$.

- <u>Examples</u>: (1) $f^*(\text{trivial}) = \text{trivial}$: if $E$ is trivial over $U \subseteq B$ then $f^*(E)$ is trivial over $f^{-1}(U) \subseteq A$.

(2) If $i: A \hookrightarrow B$, then $i^*(E)$ is the restriction of $E$ over $A \subseteq B$.

(3) Consider Möbius

$$M \longrightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$$

and let $f : S^1 \longrightarrow S^1$, $f(z) = z^2$.

Then

$$\begin{tikzcd} f^*(M) \arrow[r, "\text{pullback}"] \arrow[d, phantom, "\cong"] & M \arrow[d, phantom, "\cong"] \\ S^1 \times \mathbb{R} \arrow[r, "\text{quotient}"'] & S^1 \times \mathbb{R} / (z,t) \sim (-z,-t) \end{tikzcd}$$

so $f^*(M)$ is trivial, a 2-to-1 cover of $M$ (like paint applied to "both sides" of $M$).

• **Thm:** Suppose given a setup

$$\begin{tikzcd} & E \arrow[d, "\text{v.b.}"] \\ A \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & B \end{tikzcd}$$

with $f_0 \sim f_1$ (homotopic)

paracompact

Then $f_0^*(E) \cong f_1^*(E)$.

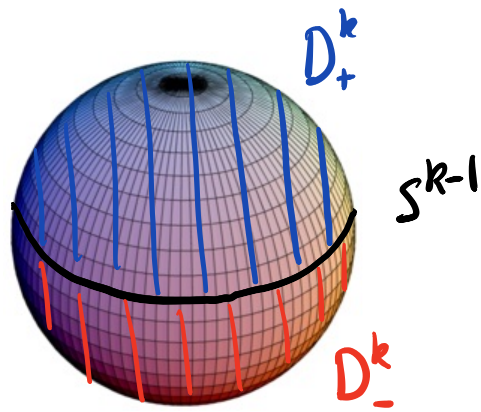
• Cor: (1) An homotopy equivalence $A \simeq B$ induces a bijection $\mathrm{Vect}^n(B) \longleftrightarrow \mathrm{Vect}^n(A)$.

(2) Any v.b. over a contractible paracompact base is trivial.

# §4 Clutching functions

- Idea: way to construct vector bundles
$$E \longrightarrow S^k.$$

- Consider $S^k = D_+^k \cup D_-^k$ union of hemispheres, with $D_+^k \cap D_-^k = S^{k-1}$.




Clutching function

- For any map $f : S^{k-1} \longrightarrow GL_n(\mathbb{F})$, we let
$$E_f = \left(D_+^k \times \mathbb{F}^n \sqcup D_-^k \times \mathbb{F}^n\right)/\sim$$
where $(x, v) \sim (x, f(x)(v))$ for $x \in S^{k-1}$, $v \in \mathbb{F}^n$.

- Now $E_f \longrightarrow S^k$ is a rank-$n$ v.b.
(can be viewed as a type of gluing construction)

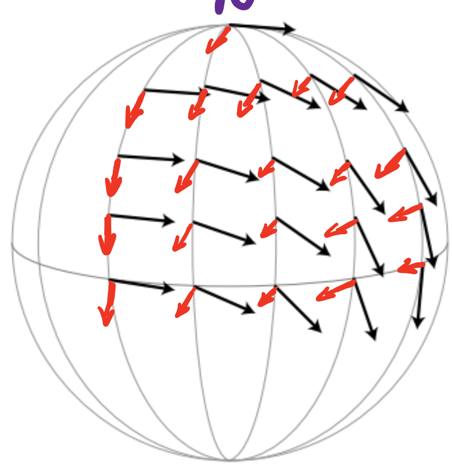
• Examples: (1) $\mathbb{F} = \mathbb{R}$, $k = 1$: the Möbius bundle $M = E_f \longrightarrow S^1$ for

$$f : S^0 = \{-1, 1\} \longrightarrow GL_1(\mathbb{R}) = \mathbb{R}^\times_0$$

$$f(-1) = -1,\ f(1) = 1.$$

(2) $\mathbb{F} = \mathbb{R}$, $k = 2$. Choose vector fields $v_+$ and $w_+$ on $D^2_+$ as follows:







We cannot extend these to $S$, but we can reflect in $z = 0$ to get $v_-$, $w_-$ on $D^2_-$.

Now let $f : S^1 \to GL_2(\mathbb{R})$ be the change of coordinates from $(v_+, w_+)$ to $(v_-, w_-)$, namely a variable rotation. Then

$$E_f \cong TS^2.$$

- **Fact:** $f \simeq g \Rightarrow E_f \cong E_g$. So if

$$[X, Y] = \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\simeq,$$

then we have

$$\Phi_{\mathbb{F}} : [S^{k-1}, GL_n(\mathbb{F})] \to \mathrm{Vect}^n(S^k).$$

- **Thm:** $\Phi_{\mathbb{C}}$ is a bijection.

**Proof:** Given $E \in \mathrm{Vect}^n(S^k)$, choose trivialisations $h_\pm : E_\pm \to D^k_\pm \times \mathbb{C}^n$.

Any two choices of $h_+$ (resp. $h_-$) differ by a map $D^k_\pm \to GL_n(\mathbb{C})$.

contractible path-connected

Hence any two choices of $h_+$ (resp. $h_-$) are homotopic, so $f = h_+ h_-^{-1} : S^{k-1} \to GL_n(\mathbb{C})$ is well defined up to homotopy

$$\rightsquigarrow \Psi(E) = [f] \in [S^{k-1}, GL_n(\mathbb{C})]$$

is well defined, a 2-sided inverse to $\Phi_{\mathbb{C}}$.

- But $GL_n(\mathbb{R})$ has two path components $GL_n^{\pm}(\mathbb{R})$...

- <u>Def</u>: An <u>orientation</u> of a v.b. $E \xrightarrow{p} B$ is an assignment of an orientation in each fibre $p^{-1}(b)$, which is mapped by some local trivialisation near $b$ to the standard orientation of $\mathbb{R}^n$.

- <u>Example</u>: Oriented line bundles over paracompact bases are trivial, because

$$s(b) = \text{positive unit vector} \in p^{-1}(b)$$

is a non-vanishing section.

Hence Möbius $M \to S^1$ is non-orientable.

- Now let
$$\mathrm{Vect}_+^n(B) = \{\text{oriented rank-}n\text{ v.b}\}/\substack{\text{oriented}\\ \text{isomorphism}}.$$
Then get clutching construction (bijection)
$$\Phi_{\mathbb{R}}: [S^{k-1}, GL_n^+(\mathbb{R})] \longrightarrow \mathrm{Vect}_+^n(S^k).$$

## References


- A. Hatcher. Vector Bundles and K-theory, Ch. 1.2. Available online.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