Suppose that a group G has an action on a set S. For variety, we shall assume that this is a right action, but totally analogous statements are also valid for left actions. For each $s \in G$ the subset of G

$$\mathrm{Stab}_G(s) = \{ g \in G \mid sg = s \}$$

is called the *stabilizer* of s in G. It is quite straightforward to observe that $1 \in \mathrm{Stab}_G(s)$, that $g^{-1} \in \mathrm{Stab}_G(s)$ whenever $x \in \mathrm{Stab}_G(s)$, and that $xy \in \mathrm{Stab}_G(s)$ whenever $x, y \in \mathrm{Stab}_G(s)$. Hence the stabilizer of S is a subgroup of G. The subset of S

$$\mathcal{O} = \{ sg \mid g \in G \}$$

is called the *orbit* of s under the action of G. If $\mathcal{O} = S$ then the action of G on S is said to be *transitive*.

As a temporary notation, for $s, t \in S$ let us write $s \sim t$ if there exists $g \in G$ such that $sg = t$. Since $s1 = s$ we have that $s \sim s$, for all $s \in S$; so the relation $\sim$ is reflexive. If $sg = t$ then $tg^{-1} = s$; thus if $s \sim t$ then $t \sim s$, and so $\sim$ is symmetric. And $\sim$ is also transitive, since if $s, t, u \in S$ with $s \sim t$ and $t \sim u$ then there exist $g, h \in G$ with $sg = t$ and $th = u$, and this yields $s \sim u$ since $s(gh) = (sg)h = th = u$. Thus $\sim$ is an equivalence relation, and in consequence the set S is the disjoint union of $\sim$-equivalence classes. The equivalence class containing s is the set $\{ t \in S \mid s \sim t \} = \{ sg \mid g \in G \}$, which is precisely the orbit of s. The orbits of G on S are the equivalence classes for the relation $\sim$ as defined above.

One can see that if the stabilizer of an element s is large then the orbit of s is small, and vice versa. The two extreme cases are as follows: if the stabilizer of s is the whole group G then the orbit is the singleton set $\{s\}$; if the stabilizer is the trivial subgroup consisting of the identity element alone, then the elements of the orbit of s are in one to one correspondence with the elements of G (since if $g, h \in G$ and $sg = sh$ then $s(gh^{-1}) = s$, which means that $gh^{-1} \in \mathrm{Stab}_G(s) = \{1\}$, and hence $g = h$). In the general case, if we write $L = \mathrm{Stab}_G(s)$ then $sg = sh$ if and only if $gh^{-1} \in L$, which is equivalent to $g \in Lh$, and this in turn is equivalent to equality of the right cosets Lg and Lh. (If we had started with a left action we would have obtained left cosets at this point: $gs = hs$ if and only if $gL = hL$.) So we conclude that there is a well defined bijective mapping $sg \mapsto Lg$ from the orbit $\mathcal{O} = \{ sg \mid g \in G \}$ to the set $\{ Lg \mid g \in G \}$ (whose elements are the right cosets in G of the stabilizer of s). Thus if $g_1, g_2, \ldots, g_m$ is a right transversal for L, so that

$$G = Lg_1 \,\dot\cup\, Lg_2 \,\dot\cup \cdots \dot\cup\, Lg_m$$

(where "$\dot\cup$" indicates disjoint union) then

$$\mathcal{O} = \{sg_1, sg_2, \ldots, sg_m\},$$

and the sg_i are pairwise distinct.

There are two different ways to define right actions of a group G on G itself. Firstly, the group's multiplication operation $G \times G \to G$ can be interpreted as a function $S \times G \to S$, where the set S is equal to G. The group axioms immediately imply that this function satisfies the defining properties of a right action. We shall call this the *right multiplication action* of G on itself. It is a transitive action—there is only one orbit—since if $s, t \in G$ are arbitrary then the element $g = s^{-1}t$ satisfies $sg = t$. Furthermore, the stabilizer of any element is trivial, since $sg = g$ implies $g = 1$. The other standard action of G on itself is the *conjugacy action*. To avoid confusion with the right multiplication action we use an exponential notation for the conjugacy action, and define

$x^g = g^{-1}xg$ for all x, $g \in G$. Note that whereas the right multiplication action is an action of G on G considered only as a set, the conjugacy action is an action of G on G considered as a group. For not only do we have $x^1 = 1^{-1}x1 = x$ and

$$x^{gh} = (gh)^{-1}x(gh) = h^{-1}(g^{-1}xg)h = (g^{-1}xg)^h = (x^g)^h,$$

for all x, g, $h \in G$, but also

$$(xy)^g = g^{-1}(xy)g = (g^{-1}xg)(g^{-1}yg) = x^g y^g$$

for all x, y, $g \in G$. The orbits of G under the conjugacy action of G are of course the conjugacy classes, as defined in Lecture 4.

Intertwining matrices

Let U and V be vector spaces over the complex field which are modules for the group G, and let $f\colon U \to V$ be a G-homomorphism. That is, f is a linear map which satisfies $g(fu) = f(gu)$ for all $u \in U$ and $g \in G$. Let $\rho\colon G \to \mathrm{GL}(V)$ and $\sigma\colon G \to \mathrm{GL}(U)$ be the representations of G on V and U respectively. That is, if $g \in G$ then ρg is the linear transformation of V given by $v \mapsto gv$ for all $v \in V$, and σg is the linear transformation of U given by $u \mapsto gu$ for all $u \in U$. For all $u \in U$ we have

$$((\rho g)f)u = (\rho g)(fu) = g(fu) = f(gu) = f((\sigma g)u) = (f(\sigma g))u,$$

and so $(\rho g)f = f(\sigma g)$. This holds for all $g \in G$. A function f which satisfies $(\rho g)f = f(\sigma g)$ is said to *intertwine* the representations ρ and σ. So here again we have two words being used to describe the same concept: an intertwining function is the same thing as a G-homomorphism.

Suppose that $u_1, u_2, \ldots, u_n$ is a basis for U and $v_1, v_2, \ldots, v_m$ is a basis for V, and let A be the matrix of f relative to these two bases. Thus A is the $m \times n$ matrix with (i,j)-entry a_{ij} satisfying $fu_j = \sum_{i=1}^{m} a_{ij}v_i$. For each $g \in G$ let $Rg \in \mathrm{GL}(m,\mathbb{C})$ be the matrix relative to the basis $v_1, v_2, \ldots, v_m$ of the transformation $v \mapsto gv$ of the space V, and let $Sg \in \mathrm{GL}(n,\mathbb{C})$ be the matrix relative to the basis $u_1, u_2, \ldots, u_m$ of the transformation $u \mapsto gu$ of the space U. So R and S are matrix versions of the representations ρ and σ. And the matrix version of the equation $(\rho g)f = f(\sigma g)$ is $(Rg)A = A(Sg)$.

Definition. If R and S are matrix representations of the group G of degrees m and n respectively then an $m \times n$ matrix A is said to *intertwine* R and S if $(Rg)A = A(Sg)$ for all $g \in G$.

So an intertwining matrix is the matrix version of a G-homomorphism.

Recall that a linear map is invertible if and only if its matrix (relative to any bases) is invertible. Of course, a matrix A can only be invertible if it is square, and this corresponds to the fact that a linear map $U \to V$ can only be invertible if U and V have the same dimension. A G-homomorphism $U \to V$ is called a *G-isomorphism* if it is invertible. The matrix version of this is an intertwining matrix which is invertible. Now if A is invertible then the equation $(Rg)A = A(Sg)$ can be rewritten as $Rg = A(Sg)A^{-1}$, and, by a definition from Lecture 3, this means that the representations R and S are equivalent. Conversely, if R and S are equivalent, so that there exists an invertible intertwining matrix A, then the linear map $f\colon U \to V$ whose matrix relative to our two fixed bases is A is a G-isomorphism. So we can say that two G-modules are G-isomorphic if and only if the corresponding matrix representations (relative to any bases) are equivalent.

Quotient modules

If S and T are arbitrary subsets of the group G then it is customary to define their product ST by the rule that $ST = \{\, st \mid s \in S,\ \text{and } t \in T \,\}$. If H is a normal subgroup of G, so that $gH = Hg$

for all $g \in G$, then $(xH)(yH) = (xy)H$ for all $x, y \in G$. This yields a well-defined multiplication operation on the set $G/H = \{ gH \mid g \in G \}$, and it can be checked that under this operation G/H is a group. The group G/H is called the *quotient* of G by H.

If the group G is Abelian (commutative) then every subgroup H is normal, and so the quotient group always exists. In particular, if V is a vector space over a field F then V is an abelian group under the operation of vector addition, and since any vector subspace U of V is also an additive subgroup of V it follows that the quotient group V/U can be formed. It is clear hat V/U is Abelian. Note that since the operation on V in this case is written as $+$, the coset of U containing the element $v \in V$ is written as $v + U$ rather than vU, and the group operation on V/U is also written as $+$. We have $V/U = \{ v + U \mid v \in V \}$,

$$(x + U) + (y + U) = (x + y) + U \qquad \text{for all } x, y \in U.$$

We now give V/U some extra structure, by defining a scalar multiplication operation on it. The relevant formula is as follows:

$$\lambda(v + U) = (\lambda v) + U \qquad \text{for all } v \in V \text{ and } \lambda \in F.$$

It is necessary to check that this is well-defined, since it is possible to have $v_1 + U = v_2 + U$ without having $v_1 = v_2$. But if $v_1 + U = v_2 + U$ then $v_1 - v_2 \in U$, and since the subspace U has to be closed under scalar multiplication it follows that $\lambda v_1 - \lambda v_2 = \lambda(v_1 - v_2) \in U$, and hence $\lambda v_1 + U = \lambda v_2 + U$. This shows that $\lambda v + U$ does not depend on the choice of the representative element v in the coset $v + U$, but only on the coset $v + U$ itself. In other words, the formula above does give a well-defined scalar multiplication operation on V/U.

Recall that a vector space over F is a set—whose elements we call "vectors"—equipped with addition and scalar multiplication operations, such that the following eight axioms are satisfied:

(i) $(u + v) + w = u + (v + w)$ for all vectors u, v and w;
(ii) $u + v = v + u$ for all vectors u and v;
(ii) there is a zero vector 0, satisfying $0 + v = v$ for all vectors v;
(iv) each vector v has a negative, which is a vector $-v$ satisfying $v + (-v) = 0$;
(v) $\lambda(\mu v) = (\lambda\mu)v$ for all scalars λ and μ and all vectors v;
(vi) $1v = v$ for all vectors v, where 1 is the identity element of F;
(vii) $\lambda(u + v) = \lambda u + \lambda v$ for all vectors u and v and all scalars λ;
(viii) $(\lambda + \mu)v = \lambda v + \mu v$ for all scalars λ and μ and all vectors v.

It is trivial to check that the addition and scalar multiplication operations we have defined on V/U satisfy these axioms. (Of course the first five of the axioms just say that a vector space is an abelian group under addition, and we had already noted above that V/U satisfies this.) It is left to the reader to check all the details. We call V/U a *quotient (vector) space*.

We proceed to embellish the above situation further by assuming that V and U are equipped with G-actions. More precisely, suppose that V is a G-module and U a submodule of V. Then the quotient space V/U is also a G-module, with G-action satisying

$$g(v + U) = (gv) + U \qquad \text{for all } g \in G \text{ and } v \in V.$$

As with addition and scalar multiplication, it is crucial to check that this G-action is well defined. The argument needed is totally analogous to the argument in the scalar multiplication case: if $v_1 + U = v_2 + U$ then $v_1 - v_2 \in U$, and since U is closed under the G action it follows that $gv_1 - gv_2 = g(v_1 - gv_2) \in U$, whence $gv_1 + U = gv_2 + U$. It is again left to the reader to check the axioms.