Modules and representations of algebras

Definition. Let F be a field and A an F-algebra. A *left F-module* is a vector space M over the field F together with a function $(a, m) \mapsto am$ from $A \times M$ to M which is bilinear and satisfies $(ab)m = a(bm)$ for all a, $b \in A$ and $m \in M$. The module M is said to be *unital* if in addition $1m = m$ for all $m \in M$, where 1 is the identity element of A.

Right modules are defined analogously, the function $A \times M \to M$ being replaced by a function $M \times A \to M$. We shall adopt the convention (which is universal) that all A-modules are assumed to be unital unless it is explicitly stated otherwise.

The connection between modules and representations works for in the same way for algebras as it does for groups. We have already defined a matrix representation of an F-algebra A to be a linear map $A \to \mathrm{Mat}_d(F)$ (for some d) which preserves multiplication and takes the identity element of A to the identity matrix. In other words, a representation of A is an algebra homomorphism $\phi: A \to \mathrm{Mat}_d(F)$ such that $\phi 1 = I$. In view of the relationship between matrices and linear transformations the following is virtually a reformulation of this definition.

Definition. A *representation* of an F-algebra A is a homomorphism ϕ from A to the algebra of all linear transformations $V \to V$, where V is a vector space over F, such that $\phi 1 = \mathrm{id}$.

We neglected to define the concept of a homomorphism of F-algebras, but the definition is obvious: an F-algebra homomorphism is a map from one F-algebra to another which preserves addition, multiplication and scalar multiplication. It is clear that the set of all linear transformations on a vector space V is an algebra if addition, multiplication and scalar multiplication of linear maps are defined in the usual way (so that multiplication of linear maps is composition). This algebra is usually denoted by $\mathrm{End}_F(V)$, since linear maps $V \to V$ are also called *F-endomorphisms* of V. The student is invited to write out for her/himself a proof of the following proposition, imitating the proof for groups given in Lecture 3.

Proposition. *Suppose that A is an F-algebra. If V is an A-module, and for each $a \in A$ we define $\phi a: V \to V$ by $(\phi a)v = av$ for all $v \in V$, then $\phi a \in \mathrm{End}_F(V)$; furthermore, the map $\phi: A \to \mathrm{End}_F(V)$, given by $a \mapsto \phi a$ for all $a \in A$, is a representation of A. Conversely, given a representation $\phi: A \to \mathrm{End}_F(V)$, the vector space V becomes an A-module if the required map $A \times V \to V$ is defined by the rule that $av = (\phi a)v$ for all $a \in A$ and $v \in V$.*

Thus if G is a group then an FG-module gives rise to a representation of the group algebra FG, and hence gives rise to a representation of G, in view of the correspondence between representations of G and representations of FG that we described in Lecture 16. So we have another item to add to the already long list of concepts that are more or less equivalent to the concept of a representation of a group!

Definition. Let A be an F-algebra. The *left regular A-module* is the vector space A made into an A-module via the map $A \times A \to A$ given by $(a, b) \mapsto ab$.

Note that this definition is in agreement with the definition, given in Lecture 9, of the regular representation of a group. The left multiplication action of G on itself can be regarded as a permutation representation of G, which in turn yields a linear representation of G on a vector space which has the elements of G as a basis. Since the group algebra FG is such a vector space, this amounts to saying that FG is a G-module, the action of G on FG being given by

$$x\Big(\sum_{g\in G}\lambda_g g\Big) = \sum_{g\in G}\lambda_g xg \tag{1}$$

for all $x \in G$ and scalars $\lambda_g \in F$. In accordance with the theorem from Lecture 16 and the discussion above, a G module is the same thing as an FG-module, and by Eq. (1) we see that in this case the map $FG \times FG \to FG$ that makes FG into an FG-module is given by

$$(\sum_{x\in G}\mu_x x, \sum_{g\in G}\lambda_g g) \mapsto \sum_{x\in G}\mu_x\Big(\sum_{g\in G}\lambda_g xg\Big) = \Big(\sum_{x\in G}\mu_x x\Big)\Big(\sum_{g\in G}\lambda_g g\Big).$$

In other words, this is the left regular FG-module.

Definition. Let A be an F-algebra. A *left ideal* in A is a submodule of the left regular module. In other words, a left ideal is a nonempty subset I of A such that

(i) $x + y \in I$ whenever $x, y \in I$,

(ii) $\lambda x \in I$ whenever $x \in I$ and $\lambda \in F$, and

(iii) $ax \in I$ whenever $a \in A$ and $x \in I$.

A left ideal I is *minimal* if $I \neq \{0\}$ and there are no nonzero left ideals J with $J \subsetneq I$. That is, the left ideal I is minimal if and only if it is an irreducible left A-module.

By Maschke's Theorem, if G is a finite group then the left regular module $\mathbb{C}G$ can be decomposed as a direct sum of minimal left ideals. To attempt to find explicitly such a direct decomposition is one possible approach to the problem of describing the irreducible representations of a finite group. We have already seen how knowledge of a full set of irreducible complex representations of G enables one to write $\mathbb{C}G$ explicitly as a direct sum of complete matrix algebras, and it is a small step from this to decompose $\mathbb{C}G$ explicitly into minimal left ideals. We shall not go into the details of this, since it is fairly straightforward, and of much less importance than the question of how to find irreducible representations. So we simply state the following result without proof, and leave it to the reader to pursue the matter or not as (s)he chooses.

Proposition. *The space F^d of d-component column vectors over the field F is an irreducible left module for the complete matrix algebra $\mathrm{Mat}_d(F)$, the map $\mathrm{Mat}_d(F) \times F^d \to F^d$ being the usual multiplication of matrices and column vectors. Furthermore, the left regular module for $\mathrm{Mat}_d(F)$ can be expressed as the direct sum of d minimal left ideals $C_1, C_2, \ldots, C_d$ which are all isomorphic to F^d. Specifically, we may take C_j to consist of those matrices whose entries in columns other than the jth column are all zero.*

A finite-dimensional F-algebra is said to be *semisimple* if the left regular module can be expressed as a direct sum of irreducible modules. The above proposition thus says that complete matrix algebras are semisimple, and it follows easily that any algebra which is a direct sum of complete matrix algebras must also be semisimple. It turns out that the converse of this is also true: a finite-dimensional algebra is which is semisimple is necessarily isomorphic to a direct sum of complete matrix algebras. We omit the proof of this.

If A is an F-algebra and $b \in A$ an arbitrary element then the set $Ab = \{\, ab \mid a \in A \,\}$ is a left ideal. It is obvious that $Ab \neq \emptyset$ (since $0b \in Ab$). Closure under addition and scalar multiplication is also clear: if $x, x' \in Ab$ and $\lambda \in F$ then there exist $a, a' \in A$ with $x = ab$ and $x' = a'b$, and this yields $x + x' = (a + a')b \in Ab$ and $\lambda x = (\lambda a)b \in Ab$. Similarly, if $x = ab \in Ab$ then for all $t \in A$ we have $tx = (ta)b \in Ab$, showing that Ab is also closed under left multiplication by elements of A.

For example, suppose that $A = \mathrm{Mat}_d(F)$ and $b \in A$ is the matrix whose entries are all zero apart from the kth diagonal entry, which is 1. That is, the (i, j)-entry of b is $\delta_{ik}\delta_{jk}$. Then if $a \in A$ is arbitrary, the (i, j)-entry of ab is $\sum_{l=1}^{d} a_{il}\delta_{lk}\delta_{jk} = a_{ij}\delta_{jk}$, which shows that ab has zero entries in all columns but the kth column, while the kth column of ab is the same as the kth column of a. So the left ideal Ab consists of all matrices which are zero in all columns but the kth. So Ab is the left ideal C_k of the proposition above. Note also that the element b is idempotent: $b^2 = b$. As we shall see, idempotent elements are of fundamental importance in representation theory. In particular, by investigating left ideals generated by idempotent elements in the group algebra of the symmetric group S_n, we shall (in the course of the next few lectures) describe a full set of irreducible $\mathbb{C}S_n$-modules.

Definition. Nonzero elements $e_1, e_2, \ldots, e_k$ in an F-algebra A form a set of *orthogonal idempotents* if $e_i^2 = e_i$ for all $i \in \{1, 2, \ldots, k\}$ and $e_ie_j = 0$ for all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$.

Definition. An idempotent e in an F-algebra A is said to be *primitive* if it is not possible to find two orthogonal idempotents e_1, e_2 with $e_1 + e_2 = e$.

The importance of primitive idempotents derives from the following proposition.

Proposition. *Let A be an F-algebra and $e \in A$ an idempotent element. Then e is primitive if and only if the left ideal Ae is indecomposable.*

Proof. Suppose first that e is not primitive; we shall show that Ae is decomposable. By our assumption there exist idempotents e_1, e_2 with $e = e_1 + e_2$ and $e_1e_2 = 0$.† Now

$$e_1 = e_1 + 0 = e_1^2 + e_1e_2 = e_1(e_1 + e_2) = e_1e,$$

and it follows that $Ae_1 = Ae_1e \subseteq Ae$. Furthermore, we also have

$$e_2 = e - e_1 = e^2 - e_1e = (e - e_1)e = e_2e,$$

whence $Ae_2 = Ae_2e \subseteq Ae$. So $Ae_1 + Ae_2 \subseteq Ae$. On the other hand, if $a \in A$ then

$$ae = a(e_1 + e_2) = ae_1 + ae_2 \in Ae_1 + Ae_2,$$

and so $Ae \subseteq Ae_1 + Ae_2$.

We have shown that $Ae = Ae_1 + Ae_2$, and since $e_i \in Ae_i$ both summands are nonzero. If we can show that $Ae_1 \cap Ae_2 = \{0\}$ then it will follow that $Ae = Ae_1 \oplus Ae_2$, and hence that Ae is decomposable. But if $x \in Ae_1$ then we have $x = ae_1$ for some $a \in A$, and therefore $x = ae_1 = ae_1^2 = (ae_1)e_1 = xe_1$. So if $x \in Ae_1 \cap Ae_2$ then $x = xe_1$ and $x = xe_2$. But substituting $x = xe_1$ into the right hand side of $x = xe_2$ gives $x = (xe_1)e_2 = x(e_1e_2) = x0 = 0$. So $Ae_1 \cap Ae_2 = \{0\}$, as required.

Conversely, suppose that Ae is decomposable. Then $Ae = I_1 \oplus I_2$ for some nonzero A-submodules I_1, I_2 of Ae. Now $e \in Ae$ and so there exist unique $x \in I_1$ and $y \in I_2$ with $e = x + y$. If $x = 0$ then $e = y \in I_2$, which implies that $Ae = Ay \subseteq I_2$ (since I_2 is a left ideal), whence

† The assumptions that e_1, e_2 and $e = e_1 + e_2$ are all idempotents and $e_1e_2 = 0$ imply that $e_2e_1 = 0$, as can be seen by expanding $(e_1 + e_2)^2$

$I_1 \oplus I_2 \subseteq I_2$, contradicting the assumption that $I_1 \neq \{0\}$. Similarly $y \neq 0$. Now since $x \in Ae$ we have $x = xe$, and thus

$$(1-x)x = x - x^2 = xe - x^2 = x(x+y) - x^2 = xy.$$

But $xy \in I_2$ since $y \in I_2$, and $(1-x)x \in I_1$ since $x \in I_1$, and since $I_1 \cap I_2 = \{0\}$ we conclude that $x - x^2 = xy = 0$. So $x = x^2$ and $xy = 0$. Exactly similar reasoning with x and y interchanged gives $y^2 = y$ and $yx = 0$. So x and y are orthogonal idempotents whose sum is e; so e is not primitive. □

The following lemma is a useful for finding idempotents in group algebras.

Lemma. *Suppose that H is a subgroup of the finite group G, and $\lambda\colon H \to \mathbb{C}^\times$ a representation of H of degree 1. Then $e = \frac{1}{|H|}\sum_{x\in H}\lambda(x^{-1})x$ is an idempotent in $\mathbb{C}G$. If $H = G$ then e is primitive.*

Proof. If $h \in H$ is fixed, then $y = hx$ runs through all elements of H as x does. So

$$\begin{aligned} he = \frac{h}{|H|}\sum_{x\in H}\lambda(x^{-1})x &= \frac{1}{|H|}\sum_{x\in H}\lambda((hx)^{-1}h)hx \\ &= \frac{1}{|H|}\sum_{y\in H}\lambda(y^{-1}h)y = \frac{1}{|H|}\sum_{y\in H}\lambda(y^{-1})\lambda(h)y = \lambda(h)e. \end{aligned}$$

It follows that

$$e^2 = \frac{1}{|H|}\sum_{h\in H}\lambda(h^{-1})he = \frac{1}{|H|}\sum_{h\in H}\lambda(h^{-1})\lambda(h)e = \frac{1}{|H|}\sum_{h\in H}\lambda(h^{-1}h)e = \frac{1}{|H|}\sum_{h\in H}e = \frac{|H|}{|H|}e = e.$$

Thus e is idempotent.

In the case $H = G$ the above calculations show that $ge = \lambda(g)e$ for all $g \in G$, and so $\bigl(\sum_{g\in G}\alpha_g g\bigr)e = \bigl(\sum_{g\in G}\alpha_g\lambda(g)\bigr)e$ for all choices of scalars α_g. So every element of $\mathbb{C}Ge$ is a scalar multiple of e. Thus the left ideal $\mathbb{C}Ge$ is a one-dimensional vector space over $\mathbb{C}$, and as $\{0\}$ is the only proper subspace of a one-dimensional space it follows that $\mathbb{C}Ge$ cannot be nontrivially expressed as a direct sum. So $\mathbb{C}Ge$ is indecomposable, and by the proposition above it follows that e is primitive. □

Note that if B is a subalgebra of A then an idempotent $e \in B$ which is primitive as an idempotent of the algebra B need not be primitive as an idempotent of A. For example, if $G = \{1, x\}$ is a cyclic group of order 2 and $H = \{1\}$ the subgroup of G of order 1, then the group algebra $\mathbb{C}H$ is a subalgebra of the group algebra $\mathbb{C}G$, and the element $1 \in \mathbb{C}H$ is a primitive idempotent of $\mathbb{C}H$. But it is not primitive as an idempotent of $\mathbb{C}G$, since it is the sum of the orthogonal idepotents $e_1 = (1/2)(1+x)$ and $e_2 = (1/2)(1-x)$. The left ideal of $\mathbb{C}H$ generated by the idempotent 1 is $\mathbb{C}H1 = \mathbb{C}H$, which is 1-dimensional, but the ideal of $\mathbb{C}G$ generated by 1 is $\mathbb{C}G1 = \mathbb{C}G$, which is two-dimensional, and the direct sum of the one-dimensional left ideals $\mathbb{C}Ge_1$ and $\mathbb{C}Ge_2$.