We have yet to prove the first part of the important theorem that was discussed in the last lecture: we have yet to prove that the space $(\widehat{X}, \widehat{d})$ is complete. Recall that we defined $\widehat{X}$ to be the quotient of the set $\mathrm{CS}(X)$ of all Cauchy sequences in $X$ by the equivalence relation $\sim$ given by the rule that $(x_n) \sim (y_n)$ if and only if $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. Thus

$$\widehat{X} = \{\, L(s) \mid s \in \mathrm{CS}(X) \,\}$$

where $L$ is the natural projection from $\mathrm{CS}(X)$ to its quotient by the above equivalence relation, and

$$L(s) = L(t) \quad \text{if and only if} \quad s = t$$

for all $s$, $t \in \mathrm{CS}(X)$. Furthermore, we showed that there is a metric $\widehat{d}$ on $\widehat{X}$ such that

$$\widehat{d}(L(s), L(t)) = \lim_{n\to\infty} d(a_n, b_n)$$

whenever $s = (a_n)_{n=1}^{\infty}$ and $t = (b_n)_{n=1}^{\infty}$ are elements of $\mathrm{CS}(X)$, and that $(X, d)$ can be identified with a subspace of $(\widehat{X}, \widehat{d})$ via an injective function $\eta\colon X \to \widehat{X}$ defined by

$$\eta(x) = L(c(x)) \quad \text{for all } x \in X,$$

where $c(x)$ is the constant sequence all of whose terms are $x$. To justify the statement that $X$ and $\eta(X)$ can be identified, we showed that $\eta$ respects distances; that is, $\widehat{d}(\eta(x), \eta(y)) = d(x, y)$ for all $x$, $y \in X$. Finally, we showed that $X$—or, more exactly, $\eta(X)$—is dense in $\widehat{X}$.

*Proof that $(\widehat{X}, \widehat{d})$ is complete.* Let $(l^{(i)})_{i=1}^{\infty}$ be a Cauchy sequence in $\widehat{X}$. Since $\eta(X)$ is dense in $\widehat{X}$ we can choose, for each positive integer $i$, an element $x_i \in X$ such that $\widehat{d}(l^{(i)}, \eta(x_i)) < (1/i)$ (for all $i \in \mathbb{Z}^+$). We show that the sequence $s = (x_i)_{i=1}^{\infty}$ is a Cauchy sequence in $X$.

Let $\varepsilon > 0$, and choose $K \in \mathbb{Z}^+$ such that $\widehat{d}(l^{(i)}, l^{(j)}) < \varepsilon/3$ for all $i$, $j > K$; such a $K$ exists since $(l^{(i)})$ is a Cauchy sequence. Now if $i$, $j > \max\{K, 3/\varepsilon\}$ then we have

$$\begin{aligned}
d(x_i, x_j) &= \widehat{d}(\eta(x_i), \eta(x_j)) && \text{(since } \eta \text{ respects distances)}\\
&\le \widehat{d}(x_i, l^{(i)}) + \widehat{d}(l^{(i)}, l^{(j)}) + \widehat{d}(l^{(j)}, x_j) && \text{(by the triangle inequality)}\\
&< \frac{1}{i} + \frac{\varepsilon}{3} + \frac{1}{j} && \text{(since } i,\ j > K)\\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && \text{(since } i,\ j > (3/\varepsilon))\\
&= \varepsilon.
\end{aligned}$$

This shows, as claimed, that $s \in \mathrm{CS}(X)$.

If $s$ is regarded as a sequence in $\widehat{X}$, it converges to the element $\alpha = L(s)$. To be more precise, the map $\eta\colon X \to \widehat{X}$ identifies $s$ with $(\eta(x_i))_{i=1}^{\infty}$, and $\eta(x_i) \to \alpha$ as $i \to \infty$ (by a lemma proved near the end of last lecture). Now

$$0 \le \widehat{d}(l^{(i)}, \alpha) \le \widehat{d}(l^{(i)}, \eta(x_i)) + \widehat{d}(\eta(x_i), \alpha) \le \frac{1}{i} + \widehat{d}(\alpha, \eta(x_i))$$

which approaches 0 as $i \to \infty$, since both terms approach 0. So the sequence $(l^{(i)})_{i=1}^{\infty}$ converges, with limit $\alpha$. But since $(l^{(i)})_{i=1}^{\infty}$ was an arbitrarily chosen Cauchy sequence in

$\widehat{X}$, it follows that all Cauchy sequences in $\widehat{X}$ are convergent. That is, $\widehat{X}$ is complete, as required. □

If $(X, d)$ and $(X', d')$ are metric spaces, a function $f: X \to X'$ is called an *isometry* if it respects distances. That is, $f$ is an isometry if $d'(f(x), f(y)) = d(x, y)$ for all $x$, $y \in X$. If, in addition, $f$ is bijective, then we say that the metric spaces $X$ and $X'$ are *isometric*. If two metric spaces are isometric, then they are essentially the same as one another: they can be regarded as two copies of the same abstract space. Thus this concept is the metric space analogue of "isomorphic" in group theory or ring theory. In the situation we considered above, the function $\eta: X \to \widehat{X}$ is an isometry, and the spaces $X$ and $\eta(X)$ are isometric to each other.

The completion of a metric space is a complete space that contains the given space as a dense subspace. Thus the following theorem says that the completion of a metric space is essentially unique. (This result was not proved in lectures, and is presented here merely for interest.)

**Theorem.** *Let $(X, d)$ and $(Y, d')$ be complete metric spaces. Suppose that $S$ and $T$ are subspaces of $X$ and $Y$ respectively, such that $S$ is dense in $X$ and $T$ is dense in $Y$. Suppose also that $S$ and $T$ are isometric. Then $X$ and $Y$ are isometric.*

*Proof.* Let $f: S \to T$ be a bijective isometry. Since $S$ is dense in $X$, for each $x \in X$ we can choose a sequence $(a_n)_{n=1}^{\infty}$ in $S$ such that $a_n \to x$ as $n \to \infty$. Since the sequence $(a_n)$ is convergent it is a Cauchy sequence, and since $f$ is an isometry it follows that the sequence $(f(a_n))_{n=1}^{\infty}$ is a Cauchy sequence in $T$, and hence in $Y$. Since $Y$ is complete this sequence must converge; that is, there exists $y \in Y$ such that $f(a_n) \to y$ as $n \to \infty$. It is not hard to see that the element $y \in Y$ depends only on the original point $x \in X$, not on the choice of the particular sequence $(a_n)$ in $S$ that converges to $x$. For, if $(b_n)$ is another such sequence then $d(a_n, b_n) \to 0$ as $n \to \infty$, and since $d'(f(a_n), f(b_n))$ (since $f$ is an isometry) it follows that

$$0 \le d'(y, f(b_n)) \le d'(y, f(a_n)) + d'(f(a_n), f(b_n)) \to 0 \quad \text{as } n \to \infty,$$

which shows that $f(b_n) \to y$ as $n \to \infty$. We conclude that there is a well-defined map $\hat{f}: X \to Y$ with the property that $\hat{f}(x) = \lim_{n\to\infty} f(a_n)$ whenever $(a_n)$ is a sequence in $S$ with $x = \lim_{n\to\infty} a_n$.

If $x$, $x' \in X$ we may choose sequences $(a_n)$, $(a'_n)$ in $S$ converging to $x$, $x'$ respectively, and we find readily that

$$d(x, x') = \lim_{n\to\infty} d(a_n, a'_n) = \lim_{n\to\infty} d'(f(a_n), f(a'_n)) = d'(\hat{f}(x), \hat{f}(x')),$$

so that $\hat{f}$ is an isometry. It is a general fact, that is left as an exercise for the reader, that isometries are necessarily injective maps. To see that $\hat{f}$ is surjective, observe that for any $y \in Y$ we may choose a sequence $(b_n)$ in $T$ with $y$ as its limit, and since $f$ is surjective there exists a sequence $(a_n)$ in $S$ such that $f(a_n) = b_n$ for all $n$. Now since $f$ is an isometry and $X$ is complete we conclude that there is an $x \in X$, such that $a_n \to x$ as $n \to \infty$, and it follows readily from the definition of $\hat{f}$ that $\hat{f}(x) = y$. □