Let $\mathcal{C}$ be the set of all continuous real-valued functions on the closed interval $[0,1]$. Let d and d' be the metrics on $\mathcal{C}$ defined as follows: for all $f,\, g \in \mathcal{C}$,

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|,$$

$$d'(f,g) = \int_0^1 |f(x) - g(x)|.$$

At the end of last lecture we posed the following two questions:
(1) Does convergence in $(\mathcal{C}, d)$ imply convergence in $(\mathcal{C}, d')$?
(2) Does convergence in $(\mathcal{C}, d')$ imply convergence in $(\mathcal{C}, d)$?

The answer to Question (1) is yes. For suppose that (f_n) converges to (f) in $(\mathcal{C}, d)$, and let $\varepsilon > 0$. Choose $N \in \mathbb{Z}$ such that $d(f_n, f) < \varepsilon$ for all $n > N$. Now whenever $n > N$ we have $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$, and so

$$\int_0^1 |f_n(x) - f(x)|\, dx < \int_0^1 \varepsilon\, dx = \varepsilon x \Big]_{x=0}^{x=1} = \varepsilon.$$

That is, $d'(f_n, f) < \varepsilon$. So we have shown that for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that $d'(f_n, f) < \varepsilon$ for all $n > N$. That is, (f_n) converges to f in $(\mathcal{C}, d')$.

The answer to Question (2) is no. In fact, convergence in $(\mathcal{C}, d')$ does not even imply pointwise convergence on $[0,1]$, as the following example illustrates. For each $n \in \mathbb{Z}^+$ we define the function f_n on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} nx & \text{if } 0 \le x \le 1/n, \\ 1 & \text{if } 1/n < x \le 1. \end{cases}$$

It is easily checked that f_n is continuous, the crucial point being that $f_n(1/n) = 1$ equals the limit of $f_n(x)$ as x approaches $1/n$ from above. Now if f is the constant function given by $f(x) = 1$ for all $x \in [0,1]$, then for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} d'(f_n, f) &= \int_0^1 |f(x) - f_n(x)|\, dx \\ &= \int_0^{1/n} |f(x) - f_n(x)|\, dx \qquad (\text{since } f(x) - f_n(x) = 0 \text{ for } 1/n \le x \le 1) \\ &= \int_0^{1/n} nx\, dx = 1/2n \longrightarrow 0 \qquad \text{as } n \to \infty. \end{aligned}$$

So the sequence (f_n) converges to f in $(\mathcal{C}, d')$. However, $f_n(0) = 0$ for all n, and so $\lim_{n\to\infty} f_n(0) = 0 \neq 1 = f(0)$. So it is not true that $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in [0,1]$. That is, (f_n) does not converge pointwise to f on $[0,1]$.

Note that pointwise convergence does not imply convergence in $(\mathcal{C}, d')$ either, as we can show with another example. Let g_n be the function defined by

$$g_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x \le 1/2n \\ n - n^2 x & \text{if } 1/2n < x \le 1/n \\ 0 & \text{if } 1/2n < x \le 1. \end{cases}$$

Then $g_n(0) = 0$ for all n, so that $\lim_{n\to\infty} g_n(x) = 0$ when $x = 0$, while for $0 < x \leq 1$ we see that $g_n(x) = 0$ whenever $n > 1/x$, and so again it follows that $\lim_{n\to\infty} g_n(0) = 0$. Thus (g_n) converges pointwise to the zero function g on $[0, 1]$. However, a short calculation shows that

$$d'(g_n, g) = \int_0^1 |g_n(x)|\, dx = 1/4$$

for all n, and so it is not true that $d'(g_n, g) \to 0$ as $n \to \infty$.

Completeness

Definition. Let (X, d) be a metric space. A sequence $(x_n)_{n=0}^{\infty}$ in X is called a *Cauchy sequence* if for all $\varepsilon > 0$ there exist an $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \varepsilon$ for all n, $m \in \mathbb{Z}$ with $n > N$ and $m > N$.

The idea is that, in some sense, the terms of a Cauchy sequence get closer and closer to each other as $n \to \infty$. Intuition might suggest that such sequences ought to converge. However, if one takes as (X, d) the set $\mathbb{Q}$ of all rational numbers with the usual metric, then one quickly sees that a Cauchy sequence in X need not have a limit in X. For example, let $x_1 = 1$, and for $n > 1$ define x_n recursively, by the formula $x_n = (x_{n-1}/2) + (1/x_{n-1})$. It is easily seen that $x_n \in \mathbb{Q}$ for all $n \in \mathbb{Z}^+$, and that $x_n \to \sqrt{2}$ as $n \to \infty$. Since $\sqrt{2} \notin \mathbb{Q}$ the sequence (x_n) does not have a limit in $\mathbb{Q}$. Since it does have a limit in $\mathbb{R}$ (with the usual metric) it is not hard to show that (x_n) is a Cauchy sequence. The metric space $\mathbb{Q}$ is incomplete, in the sense that it does not have limits for all its Cauchy sequences.

Definition. A metric space X is said to be *complete* if every Cauchy sequence in X has a limit in X.