THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 12

1. (*i*) Suppose that X is a connected space, and S a nonempty proper subset of X. Prove that the frontier of S is nonempty.

(*ii*) Suppose that X is a disconnected space. Prove that X has a nonempty proper subset S whose frontier is empty.

Solution.

(*i*) The frontier of S is $\overline{S} \setminus \mathrm{Int}(S)$. If this is empty then $\overline{S} \subseteq \mathrm{Int}(S)$. Since we also have $\mathrm{Int}(S) \subseteq S$ and $S \subseteq \overline{S}$, it follows that $\mathrm{Int}(S) = S = \overline{S}$. So S is a nonempty proper subset of X that is both open and closed. Hence X is disconnected.

(*ii*) Given that X is disconnected, there exists a nonempty proper subset S of X that is both open and closed. So $\mathrm{Int}(S) = S = \overline{S}$, and hence $\overline{S} \setminus \mathrm{Int}(S) = \emptyset$. That is, the frontier of S is empty.

2. Recall that a topology on a set X is a collection $\mathcal{T}$ of subsets of X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, and $\emptyset$, $X \in \mathcal{T}$. (The sets in the collection $\mathcal{T}$ are called the open sets of the topology.) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$, and $\mathcal{T}_2$ finer than $\mathcal{T}_1$, if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. (That is, all open sets of the coarser topology are also open in the finer topology.)

If X is connected in some topology, is it necessarily connected in a finer topology? Is it necessarily connected in a coarser topology?

Solution.

The finest possible topology on X is the discrete topology, in which every set is open. Provided X has at least two points it will be disconnected in the discete topology, since $X = \{x\} \cup (X \setminus \{x\})$ (where x is any element of X) expresses X as the disjoint union of two nonempty open sets. So it is certainly not true that a connected space remains connected when we pass to a finer topology.

If a space is connected then it cannot be expressed as the disjoint union of two nonempty open sets. If we pass to a coarser topology then there will be fewer open sets available. Every set that is open in the coarser topology was already open in the finer topology; so any disconnection in the coarser topology would have already been a disconnection in the finer topology. So a connected set remains connected when moving to a coarser topology.

3. Prove that if X and Y are homeomorphic topological spaces and X is compact then Y is compact.

Solution.

A homeomorphism is a bijective function f such that f and f^{-1} both take open sets to open sets. (Note that if S is a subset of the codomain of f then by definition, $f^{-1}(S) = \{\, x \mid f(x) \in S \,\}$. This is defined even if there is no function f^{-1} inverse to f. But when f is bijective, so that the inverse function does exist, $f^{-1}(S)$ coincides with $\{\, f^{-1}(s) \mid s \in S \,\}$.) Any property whose definition can be phrased in terms of open sets will necessarily be preserved by homeomorphisms. Since compactness is such a property, it must be preserved. Furthermore, the proof should be routine: write out what it means to say that X is compact, apply the homeomorphism to transfer everything from X to Y, and the result will be a statement that says that Y is compact.

A space is compact if every open covering of the space has a finite subcovering. Assume that X has this property, and that $f\colon X \to Y$ is a homeomorphism. We must prove that every open covering of Y has a finite subcovering. So, suppose that $(V_i)_{i\in I}$ is a family of open subsets of Y such that $Y = \bigcup_{i\in I} V_i$. For each $i \in I$ the set $f^{-1}(V_i)$ is an open subset of X, since f^{-1} takes open sets to open sets. If $x \in X$ then $f(x) \in Y = \bigcup_{i\in I} V_i$; so $f(x) \in V_i$ for some $i \in I$, and so $x \in f^{-1}(V_i)$ for some $i \in I$. Thus $X = \bigcup_{i\in I} f^{-1}(V_i)$, whence $(f^{-1}(V_i))_{i\in I}$ is an open covering of X, and since X is compact there is a finite subset J of I such that $X = \bigcup_{i\in J} f^{-1}(V_i)$.

Now if $y \in Y$ is arbitrary then $f^{-1}(y) \in X = \bigcup_{i\in J} f^{-1}(V_i)$, and therefore $f^{-1}(y) \in f^{-1}(V_i)$ for some $i \in J$. It follows that

$$y = f(f^{-1}(y)) \in f(f^{-1}(V_i)) = V_i \qquad \text{(for some } i \in J\text{)},$$

and since y was an arbitrary element of Y, this shows that $(V_i)_{i\in J}$ is a covering of Y. But J is a finite subset of I, and $(V_i)_{i\in I}$ was originally chosen as an arbitrary open covering of Y. So we have shown that every open covering of Y has a finite subcovering; that is, Y is compact.

(Note that in lectures we proved the stronger statement that the continuous image of any compact set is compact. The proof was very similar to that given above, which in fact does not use all aspects of the assumption that f is a homeomorphism.)

4. Find a compact topological space X such that for some $x \in X$ the subspace $X \setminus \{x\}$ of X is homeomorphic to $\mathbb{R}$.

Solution.

It can be shown that $\mathbb{R}$ is homeomorphic to each open interval (a, b). The closed interval $[a, b]$ is compact; so one ought to be able to get a compact space by adding two extra points to $\mathbb{R}$. But one extra point suffices, because we can identify a and b. Geometrically, this corresponds to wrapping the open interval round a circle, so that the ends come up against each other. The addition of one extra point then makes a circle, which is a compact subset of $\mathbb{R}^2$.

This can be done totally explicitly by use of the stereographic projection (as described on p. 60 of Choo's notes.), which gives a homeomorphism from $\mathbb{R}$ to the set $X = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \} \setminus \{(0,1)\}$. The formula is $x \mapsto (2x/(x^2+1), (x^2-1)/(x^2+1))$, and the inverse mapping $X \to \mathbb{R}$ is given by $(x_1, x_2) \mapsto x_1/(1-x_2)$. Thus the unit circle $S \subseteq \mathbb{R}^2$ is a compact space such that $S \setminus \{0, 1\}$ is homeomorphic to $\mathbb{R}$.

It is interesting that it is possible to generalize this result. We have shown that by adding an extra point—let us call it ∞—to $\mathbb{R}$, we can produce a compact space $\mathbb{R} \cup \{\infty\}$ (homeomorphic to the unit circle S). What are the open sets of this new space?

If d is the usual distance function on $\mathbb{R}^2$ then, for each point $(a_1, a_2) \in S$ and $\varepsilon \in \mathbb{R}$, the set $\{ (x_1, x_2) \in S \mid d((x_1, x_2), (a_1, a_2)) < \varepsilon \}$ is an open subset of S. Let us call such sets "open arcs". Just as every open subset of $\mathbb{R}$ is a union of open intervals, so every open subset of S is a union of open arcs. We can easily identify the subsets of $\mathbb{R} \cup \{\infty\}$ that correspond to open arcs. Firstly, an open arc that does not contain $(0, 1)$ simply corresponds to an open interval in $\mathbb{R}$. An open arc that contains $(0, 1)$ corresponds to a subset of $\mathbb{R} \cup \{\infty\}$ of the form $(-\infty, \alpha) \cup \{\infty\} \cup (\beta, \infty)$, for some α, $\beta \in \mathbb{R}$ with $\alpha < \beta$. Observe that the complement of this set is the closed and bounded interval $[\alpha, \beta]$. An arbitrary open subset of $\mathbb{R} \cup \{\infty\}$ is a union of subsets corresponding to open arcs. If none of these arcs contain $(0, 1)$ the result is simply an open subset of $\mathbb{R}$, and we can obtain any open subset of $\mathbb{R}$ in this way. If any of the arcs contain $(0, 1)$ then we obtain a subset of $\mathbb{R} \cup \{\infty\}$ that is the union of $\{\infty\}$ and a collection of open intervals in $\mathbb{R}$, of which at least one has the form $(-\infty, \alpha)$ and at least one has the form (β, ∞). This results in a set of the form $\{\infty\} \cup U$, where U is an open subset of $\mathbb{R}$ whose complement is bounded, and any such set can be obtained. The complement of an open set is closed, and by the Heine-Borel Theorem a closed subset of $\mathbb{R}$ that is bounded is compact; so we can summarize what we have shown as follows: the open sets of $\mathbb{R} \cup \{\infty\}$ are the open sets of $\mathbb{R}$ together with sets of the form $\{\infty\} \cup (\mathbb{R} \setminus C)$, where C is a compact subset of $\mathbb{R}$.

Phrased like this, the same construction works for any topological space X in place of $\mathbb{R}$. Add a new point, ∞, to X, and define a topology on $X \cup \{\infty\}$ to consist of all open subsets of X together with all sets of the form $\{\infty\} \cup (X \setminus C)$, where C is a compact subset of X. This space $X \cup \{\infty\}$ is compact and the subspace obtained by removing ∞ is the original space X. (The proof of this is left as a challenge for the keen student.)

5. Prove that there is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ with the property that for each $t \in \mathbb{R}$ the equation $f(x) = t$ has exactly two solutions x.

Solution.

This is an exercise in the use of the Intermediate Value Theorem. Suppose

such an f exists. Then there must exist a, $b \in \mathbb{R}$ with $a < b$ and $f(a) = f(b)$. If $c \in (a, b)$ then $f(c) \neq f(a)$, or else $f(x) = f(a)$ would have at least three solutions. Replacing f by $-f$ if need be, we may suppose that $f(c) > f(a)$. The restriction of f to the compact interval $[a, b]$ must achieve a maximum value $M \geq f(c)$ on $[a, b]$. Suppose that $f(d) = M$, where $d \in (a, b)$. By the assumption about f there must be two solutions of $f(x) = M + 1$. Choose e with $f(e) = M + 1$. Then $e \notin [a, b]$, since $f(e) > M = \max_{x \in [a,b]} f(x)$. Suppose $e < a$. Applying the Intermediate Value Theorem on each of the intervals $[e, a]$, $[a, d]$ and $[d, b]$ we see that $f(x) = \frac{1}{2} f(a) + M$ must have solutions in each of (e, a), (a, d) and (d, b) contradicting the fact that this equation should have exactly two solutions. A similar contradiction is obtained if $e > b$.

6. Show that the initial value problem $x'(t) = 3x(t)^{2/3}$, with $x(0) = 0$, has infinitely many solutions x, by showing that

$$x(t) = \begin{cases} 0 & \text{if } t < c \\ (t-c)^3 & \text{if } t \geq c \end{cases}$$

is a solution for all $c > 0$. Reconcile this with Picard's Theorem.

Solution.

On subintervals of $(-\infty, c)$ the function $x(t)$ is zero, and so its derivative is $0 = 0^{2/3} = x(t)^{2/3}$. We have $x'(t) = 3(t-c)^2 = 3x(t)^{2/3}$ on subintervals of (c, ∞). The initial condition is obviously satisfied; so it remains to check that the differential equation is satisfied at $t = c$. But

$$\lim_{t \to c^+} \frac{x(t) - x(c)}{t - c} = \lim_{t \to c^+} (t - c)^2 = 0 = \lim_{t \to c^-} \frac{x(t) - x(c)}{t - c};$$

so $x'(c)$ exists and equals 0, which is also $x(c)^{2/3}$, as required.

Picard's Theorem says that under suitable conditions initial value problems have unique solutions. Here the solution is not unique; so evidently the hypotheses of Picard's Theorem are not satisfied. The theorem as we proved it in lectures applies only to differential equations of the form $x'(t) = f(t, x)$, where the function f satisfies a Lipschitz condition with respect to x. Here $f(t, x) = x^{2/3}$, and this does not satisfy a Lipschitz condition on any interval $(-\varepsilon, \varepsilon)$, since $(x^{2/3} - 0)/(x - 0) \to \infty$ as $x \to 0^+$.

7. Let S be the two element set $\{0, 1\}$, and give S the discrete topology. (That is, all four subsets are open.) Prove that a topological space X is connected if and only if there is a continuous surjection $X \to S$.

Solution.

The idea is that if $f\colon X \to S$ is continuous and surjective then $U_1 = f^{-1}(\{0\})$ and $U_2 = f^{-1}(\{1\})$ provide a disconnection of X; conversely, given a disconnection $X = U_1 \cup U_2$ we can define a continuous surjection f by letting it take the value 0 on U_1 and 1 on U_2. See Choo's notes or the web version of Lecture 22 for a more detailed proof.