THE UNIVERSITY OF SYDNEY

PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 9

(For all subspaces of $\mathbb{R}$, use the usual (Euclidean) metric.)

1. Let $X = (0, 1/4)$ and let $f: X \to X$ be given by $f(x) = x^2$. Prove that f is a contraction mapping with no fixed point in X. Reconcile this with the Contraction Mapping Theorem.

Solution.

A closed subset of a complete space is a complete space, but $(0, 1/4)$ is not closed in $\mathbb{R}$. And indeed it is not complete, since (for example) the sequence $(1/n)_{n=1}^{\infty}$ is a Cauchy sequence in $(0, 1/4)$ which has no limit in $(0, 1/4)$. Completeness of the space is an important hypothesis of the Contraction Mapping Theorem; since the hypotheses are not all satisfied here, the theorem cannot be applied.

If $0 < x < 1/4$ then $0 < x^2 < 1/16 < 1/4$; hence $f(x) = x^2$ does define a map from $(0, 1/4)$ to $(0, 1/4)$. The only solutions in $\mathbb{R}$ of $x^2 = x$ are $x = 0$ and $x = 1$, neither of which lie in $(0, 1/4)$. So f has no fixed points. Finally, if $x, y \in (0, 1/4)$ then

$$|x^2 - y^2| = |x - y|\,|x + y| \le |x - y|(|x| + |y|) \le |x - y|(\tfrac{1}{4} + \tfrac{1}{4});$$

that is, $d(f(x), f(y)) \le \frac{1}{2} d(x, y)$, which shows that f is a contraction mapping.

2. Let $X = \{\, x \in \mathbb{Q} \mid x \ge 1 \,\}$ and let $f: X \to X$ be defined by $f(x) = \frac{x}{2} + \frac{1}{x}$. Show that f is a contraction mapping and that f has no fixed point in X.

Solution.

The question asserts that given formula for f defines a mapping $X \to X$; let us check that it is true. If $x \in \mathbb{Q}$ then $x/2 \in \mathbb{Q}$ and $1/x \in \mathbb{Q}$; so $(x/2) + (1/x) \in \mathbb{Q}$. If $1 \le x \le 2$ then $1/2 \le 1/x \le 1$ and $1/2 \le x/2 \le 1$; so $(x/2) + (1/x) \ge (1/2) + (1/2) = 1$. If $x > 2$ then $(x/2) + (1/x) > x/2 > 1$. So if $x \in \mathbb{Q}$ and $x \ge 1$ then $(x/2) + (1/x) \ge 1$, as required.

If $f(x) = x$ then $(x/2) + (1/x) = x$, which gives $1/x = x/2$, and $x = \pm\sqrt{2}$. So $f(x) = x$ has no solution in X. And if $x, y \in X$ then

$$d(f(x), f(y)) = |(x - y)(\tfrac{1}{2} - \tfrac{1}{xy})| \le \tfrac{1}{2} d(x, y)$$

since $x, y \ge 1$ gives $-\frac{1}{2} \le \frac{1}{2} - \frac{1}{xy} < \frac{1}{2}$. So f is a contraction mapping. (Again the space X is not complete.)

3. Let $X = [1, \infty)$ and let $f: X \to X$ be given by $f(x) = x + 1/x$. Show that $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ with $x \ne y$, and show that f has no fixed point in X.

Solution.

This time the Contraction Mapping Theorem will not apply since it is not in fact true that there is an $\alpha < 1$ with $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$. (So f is not a contraction mapping in the sense of the theorem.) It is clear that $x + (1/x) \ge 1$ whenever $x \ge 1$ (since $1/x > 0$), and so the formula does define a function $X \to X$. There is no fixed point, since $x + (1/x) = x$ gives $1/x = 0$, and hence $1 = 0$. Now for $x, y \ge 1$,

$$d(f(x), f(y)) = |(x - y)(1 - \tfrac{1}{xy})| = d(x, y)(1 - \tfrac{1}{xy}) < d(x, y),$$

as claimed.

4. Let $X = [1, \infty)$ and let $f: X \to X$ be given by $f(x) = \frac{25}{26}(x + 1/x)$. Show that $d(f(x), f(y)) \le \frac{25}{26} d(x, y)$ (whence f is a contraction mapping). By solving the equation algebraically, show that 5 is the unique fixed point of f.

Solution.

$d(f(x), f(y)) = \frac{25}{26} d(x, y)(1 + \frac{1}{xy}) \le \frac{25}{26} d(x, y)$ (cf. Question 3). It is important that the given formula does define a map $X \to X$; so let us check that $x \ge 1$ implies $\frac{25}{26}(x + (1/x)) \ge 1$. If $x \ge \frac{26}{25}$ then $\frac{25}{26}(x + (1/x)) > \frac{25}{26} x \ge 1$. If $1 \le x \le \frac{26}{25}$ then $\frac{25}{26} \le \frac{1}{x} \le 1$ and $1 + \frac{25}{26} \le x + \frac{1}{x}$, giving $\frac{25}{26}(x + \frac{1}{x}) \ge \frac{1275}{676} > 1$. Note that X is a complete metric space, being a closed subset of the complete space $\mathbb{R}$. So the Contraction Mapping Theorem guarantees that there is a unique $x \in X$ with $f(x) = x$. In this case we can easily confirm this by just solving the equation. If $f(x) = x$ then $\frac{26}{25} x = x + \frac{1}{x}$, giving $x^2 = 25$, so that $x = 5$ is the only possible solution in X. (And $f(5) = 5$ is easily checked.)

5. Let $X = [0, 1]$ and let $f: X \to X$ be given by $f(x) = \frac{1}{7}(x^3 + x^2 + 1)$. Show that $d(f(x), f(y)) \le \frac{5}{7} d(x, y)$. Calculate $f(0)$, $f^{(2)}(0)$, $f^{(3)}(0)$, ..., and hence find, to three decimal places, the fixed point of f.

Solution.

If $0 \le x \le 1$ then $0 \le x^3 + x^2 + 1 \le 3$, and so $0 \le \frac{1}{7}(x^3 + x^2 + 1) \le \frac{3}{7} < 1$. So again the question has not lied: we do have a function $X \to X$. Now

$$d(f(x), f(y)) = \tfrac{1}{7}|(x^3 - y^3) + (x^2 - y^2)| = \tfrac{1}{7} d(x, y)|x^2 + xy + y^2 + x + y| \le \tfrac{5}{7} d(x, y)$$

whenever $x, y \in [0, 1]$ (since x^2, xy etc. are all in $[0, 1]$). Again X is a complete space, being a closed subset of $\mathbb{R}$. According to my calculator: $f(0) \approx 0.142857$, $f^{(2)}(0) \approx 0.146189$, $f^{(3)}(0) \approx 0.1463565$, $f^{(4)}(0) \approx 0.1463650$, ... —but this is not a course in numerical analysis.

6. Let $X = [1,2] \cap \mathbb{Q}$ and let $f\colon X \to X$ be defined by $f(x) = -\frac{1}{4}(x^2-2)+x$. Prove that f is a contraction mapping and that f has no fixed point in X.

Solution.

The space is not complete, of course. If X were defined simply to be $[1,2]$ then the Contraction Mapping Theorem would apply; however, the solution of $f(x) = x$ in $[1,2]$ turns out to be irrational, and so not in the set X as actually defined. Indeed, $f(x) = x$ if and only if $x^2 - 2 = 0$, and this has no solution in $\mathbb{Q}$. The quadratic $-\frac{1}{4}(x^2-2)+x$ has its turning point at 2, where the function value is $\frac{3}{2}$. So $f(1) = \frac{5}{4} \le f(x) \le f(2) = \frac{3}{2}$ for all $x \in [1,2]$. This confirms that $f(x) \in [1,2]$, as the question asserts. And it is also clear that $x \in \mathbb{Q}$ implies $-\frac{1}{4}(x^2-2)+x \in \mathbb{Q}$. For all x, $y \in [1,2]$,

$$d(f(x), f(y)) = |x-y|\,|1 - \tfrac{1}{4}(x+y)| \le \tfrac{1}{2}d(x,y)$$

(since $-\frac{1}{4} \le 1 - \frac{1}{4}(x+y) \le \frac{1}{2}$); so f is a contraction.

7. Let $f\colon [a,b] \to [a,b]$ be differentiable over $[a,b]$. Show that f is a contraction mapping if and only if there exists a number $K < 1$ such that $|f'(x)| \le K$for all $x \in (a,b)$.

Solution.

Suppose first of all that f is a contraction mapping. Then there exists $K < 1$ such that $|f(x) - f(y)| \le K|x-y|$ for all x, $y \in [a,b]$. So if $x \in (a,b)$ is arbitrary, then

$$f'(x) = \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \le \lim_{y \to x} K = K.$$

Conversely, suppose that $K < 1$ and $|f'(x)| \le K$ for all $x \in (a,b)$. Then for arbitrary x, $y \in [a,b]$ we have that $f(x) - f(y) = f'(t)(x-y)$ for some $t \in (x,y) \subseteq (a,b)$ (by the Mean Value Theorem). This gives

$$|f(x) - f(y)| = |f'(t)|\,|x-y| \le K|x-y|,$$

as required.

8. Let $\mathcal{C}$ be the set of continuous functions $[0,1] \to \mathbb{R}$, and let d be given by $d(f,g) = \sup_{x \in [0,\,1]} |f(x) - g(x)|$. Define $F\colon \mathcal{C} \to \mathcal{C}$ by $(Ff)(x) = \int_0^x f(t)\,dt$ (for all $f \in \mathcal{C}$). Show that for all f, $g \in X$ and all $x \in [0,1]$,

(*i*) $(Ff)(x) - (Fg)(x) \le x\,d(f,g)$, and

(*ii*) $(F^{(2)}f)(x) - (F^{(2)}g)(x) \le \frac{x^2}{2}d(f,g)$,

and deduce that $F^{(2)}$ is a contraction mapping. Show, however, that F is not a contraction mapping.

Solution.

(*i*) $(Ff)(x) - (Fg)(x) = \int_0^x (f(t) - g(t))\,dt \le \int_0^x d(f,g)\,dt \le x\,d(f,g)$.

(*ii*)
$$\begin{aligned}(F^{(2)}f)(x) - (F^{(2)}g)(x) &= \int_0^x ((Ff)(t) - (Fg)(t))\,dt \\ &\le \int_0^x t d(f,g)\,dt = \tfrac{x^2}{2}\,d(f,g).\end{aligned}$$

Thus $d(Ff, Fg) = \sup_{x \in [0,\,1]} |(Ff)(x) - (Fg)(x)| \le \sup_{x \in [0,1]} x\,d(f,g) \le d(f,g)$, and therefore

$$d(F^{(2)}f, F^{(2)}g) = \sup_{x \in [0,1]} |(F^{(2)}f)(x) - (F^{(2)}g)(x)| \le \sup_{x \in [0,1]} \tfrac{x^2}{2} d(f,g) \le \tfrac{1}{2}d(f,g).$$

Hence $F^{(2)}$ is a contraction mapping. But F is not, since by taking $f = 1$ and $g = 0$, we have $d(Ff, Fg) = \sup_{x \in [0,1]} x = 1 = d(f,g)$.

9. (*Square sum criterion*) Show that if the $n \times n$ matrix C (over $\mathbb{R}$) satisfies $\sum_{j=1}^n \sum_{k=1}^n c_{jk}^2 < 1$, then for any $b \in \mathbb{R}^n$ the linear system $x = Cx + b$ has a unique solution. (Imitate the proof of Theorem 2.1 of Choo's notes, using the Euclidean metric ($d(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$) instead of the one used there.)

Solution.

Define $f\colon \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = Cx + b$. Let x, $y \in \mathbb{R}^n$ and put $z = f(x)$ and $w = f(y)$. Then $z - w = C(x-y)$, and so $z_i - w_i = \sum_{j=1}^n c_{ij}(x_j - y_j)$ for each i. Apply the Cauchy-Schwarz inequality (which says that the dot product of two vectors in $\mathbb{R}^n$ is at most the product of their lengths—here the two vectors in question are the i-th row of C and $x - y$). We deduce that $|z_i - w_i| \le \sqrt{\sum_{j=1}^n c_{ij}^2}\sqrt{\sum_{j=1}^n (x_j - y_j)^2}$. Squaring and summing on i gives $\sum_{i=1}^n (z_i - w_i)^2 \le \left(\sum_{i=1}^n \sum_{j=1}^n c_{ij}^2\right)\left(\sum_{j=1}^n (x_j - y_j)^2\right)$. That is, $d(f(x), f(y)) \le K d(x,y)$, where $K = \sum_{i=1}^n \sum_{j=1}^n c_{ij}^2 < 1$. Thus f is a contraction mapping, and so has a unique fixed point.