THE UNIVERSITY OF SYDNEY
PURE MATHEMATICS 3901

Metric Spaces 2000

Tutorial 5

1. Let $X = (X, d)$ be a metric space. Let (x_n) and (y_n) be two sequences in X such that (y_n) is a Cauchy sequence and $d(x_n, y_n) \to 0$ as $n \to \infty$. Prove that

(*i*) (x_n) is a Cauchy sequence in X, and

(*ii*) (x_n) converges to a limit x if and only if (y_n) also converges to x.

Solution.

(*i*) Let $\varepsilon > 0$. Since $d(x_n, y_n) \to 0$ as $n \to \infty$, there is N_1 such that $d(x_k, y_k) < \varepsilon/3$ for all $k > N_1$. Since (y_n) is a Cauchy sequence, there is N_2 such that $d(y_m, y_n) < \varepsilon/3$ for all m, $n > N_2$. Put $N = \max\{N_1, N_2\}$. Then, by the triangle inequality, for all m, $n > N$ we have

$$d(x_m, x_n) \le d(x_m, y_m) + d(y_m, y_n) + d(y_n, x_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence (x_n) is a Cauchy sequence.

(*ii*) Suppose that (y_n) converges to x. Then $d(y_n, x) \to 0$ as $n \to \infty$. Now by the triangle inequality,

$$0 \le d(x_n, x) \le d(x_n, y_n) + d(y_n, x) \longrightarrow 0 + 0 = 0$$

as $n \to \infty$; so $d(x_n, x) \to 0$ as $n \to \infty$. So (x_n) converges to x. Similarly, if (x_n) converges to x then $0 \le d(y_n, x) \le d(y_n, x_n) + d(x_n, x) \to 0$ as $n \to \infty$, whence (y_n) converges to x also.

2. Prove that every Cauchy sequence in a metric space (X, d) is bounded.

Solution.

(This was proved in lectures). Let (x_n) be a Cauchy sequence of (X, d). By the definition of Cauchy sequence, applied with $\varepsilon = 1$, there exists N such that $d(x_m, x_n) < 1$ for all m, $n \ge N$; so $x_n \in B(x_N, 1)$ for all $n \ge N$. Now define $r = 1 + \max\{1, d(x_1, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N)\}$. We see that $x_n \in B(x_N; r)$ for all n; so (x_n) is bounded.

3. Show that the set X of all integers, with metric d defined by $d(m, n) = |m - n|$, is a complete metric space.

Solution.

Note that d is the metric induced by the Euclidean metric (the usual metric) on $\mathbb{R}$. Since closed subspaces of complete spaces are complete, it suffices to show that $\mathbb{Z}$ is closed in $\mathbb{R}$. The complement of $\mathbb{Z}$ in $\mathbb{R}$ is the union of all the open intervals $(n, n+1)$, where n runs through all of $\mathbb{Z}$, and this is open since every union of open sets is open. So $\mathbb{Z}$ is closed.

Alternatively, let (a_n) be a Cauchy sequence in $\mathbb{Z}$. Choose an integer N such that $d(x_n, x_m) < 1$ for all $n \ge N$. Put $x = x_N$. Then for all $n \ge N$ we have $|x_n - x| = d(x_n, x_N) < 1$. But x_n, $x \in \mathbb{Z}$, and since two distinct integers always differ by at least 1 it follows that $x_n = x$. This holds for all $n > N$. So $x_n \to x$ as $n \to \infty$ (since for all $\varepsilon > 0$ we have $0 = d(x_n, x) < \varepsilon$ for all $n > N$).

4. (*i*) Show that if D is a metric on the set X and $f\colon Y \to X$ is an injective function then the formula $d(a, b) = D(f(a), f(b))$ defines a metric d on Y, and use this to show that $d(m, n) = |m^{-1} - n^{-1}|$ defines a metric on the set $\mathbb{Z}^+$ of all positive integers.

(*ii*) Show that $(\mathbb{Z}^+, d)$, where d is as defined in Part (*i*), is not a complete metric space.

Solution.

(*i*) This is obvious, since we can regard f as identifying Y with X. Nevertheless, let us write out the details. If a, b, $c \in Y$, then $f(a)$, $f(b)$, $f(c) \in X$. Since D is a metric on X, we have

$$D(f(b), f(c)) \le D(f(a), f(b)) + D(f(a), f(c))$$

and

$$D(f(a), f(b) = D(f(b), f(a)) \ge 0 \quad \text{with equality only if } f(a) = f(b).$$

Thus for all a, b, $c \in Y$,

$$d(b, c) = D(f(b), f(c)) \le D(f(a), f(b)) + D(f(a), f(c)) = d(a, b) + d(a, c),$$

which shows that d satisfies the triangle inequality. Similarly, for all a, $b \in Y$

$$d(a, b) = D(f(a), f(b) = D(f(b), f(a)) = d(a, b),$$

and

$$d(a, b) = d(f(a), f(b)) \ge 0 \quad \text{with equality only if } f(a) = f(b).$$

Since f is injective, $f(a) = f(b)$ if and only if $a = b$; so we deduce that $d(a, b) = d(b, a) \ge 0$ with equality only if $a = b$, as required.

The astute reader will have noticed that it was necessary only to assume that f is injective, rather than bijective.

The function $f\colon \mathbb{Z}^+ \to \mathbb{R}$ defined by $f(n) = n^{-1}$ for all $n \in \mathbb{Z}^+$ is certainly injective, and if we take D to be the usual metric on $\mathbb{R}$ and apply the principle we have been discussing, we obtain that

$$d(m, n) = D(f(m), f(n)) = D(m^{-1}, n^{-1}) = |m^{-1} - n^{-1}|$$

defines a metric on $\mathbb{Z}$, as claimed. (Or, observe that $n \to n^{-1}$ gives a bijection from $\mathbb{Z}^+$ to $\{ n^{-1} \mid n \in \mathbb{Z}^+ \}$, which has a metric induced from the usual metric on $\mathbb{R}$.)

(*ii*) The sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n = n$ is a Cauchy sequence with respect to the metric described in Part (*i*). To see this, let $b_n \in \mathbb{R}$ be defined by $b_n = f(a_n) = n^{-1}$ for all $n \in \mathbb{Z}^+$. Since (b_n) is a convergent sequence in $\mathbb{R}$ (with limit 0), it is a Cauchy sequence. Furthermore, since $d(a_n, a_m) = D(f(a_n), f(a_m)) = D(b_n, b_m)$ for all $n, m \in \mathbb{Z}^+$, the fact that (b_n) is Cauchy implies that (a_n) is Cauchy also.

Of course, a direct proof is trivial: given $\varepsilon > 0$, if we define $N = 1/\varepsilon$ then it follows that $n^{-1}, m^{-1} \in (0, \varepsilon)$, and so $|n^{-1} - m^{-1}| < \varepsilon$, for all $n, m > N$.

5. Let c be the set of all sequences $x = (x_k)$ of complex numbers that are convergent in the usual sense, and let d be the metric on c induced from the space ℓ^∞. (That is, $d(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|$). Show that the metric space (c, d) is complete. [Hint: Show that c is closed in ℓ^∞.]

Solution.

Since C is complete, a sequence in $\mathbb{C}$ is convergent if and only if it is a Cauchy sequence. So c can be described as the set of all Cauchy sequences in $\mathbb{C}$. Recall that ℓ^∞ is the set of all bounded sequences in $\mathbb{C}$, with the sup metric. Every Cauchy sequence is bounded; so (c, d) is indeed a subspace of ℓ^∞. The space ℓ^∞ is complete, by Example 2.6 on p. 41 of Choo's notes. Since a closed subspace of a complete space is complete, it suffices to show that c is a closed subset of ℓ^∞. So it suffices to show that $\overline{c} \subseteq c$.

Let $x \in \overline{c}$. Then there exists a sequence $(x^{(k)})_{k=1}^{\infty}$ of points of c converging in ℓ^∞ to the point x. Our task is to prove that $x \in c$. Since points of ℓ^∞ are themselves sequences, let us write $x_i^{(k)}$ for the i-th term of $x^{(k)}$ and x_i for the i-th term of x. That is,

$$\begin{aligned} x^{(1)} &= (x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots), \\ x^{(2)} &= (x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots), \\ x^{(3)} &= (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \ldots), \\ \cdots & \quad \cdots \quad \cdots \\ x &= (x_1, x_2, x_3, \ldots). \end{aligned}$$

We are given that each $x^{(k)}$ is a Cauchy sequence, and the aim is to prove that x is a Cauchy sequence. We are also given that $(x^{(k)})$ converges in the ℓ^∞ metric—that is, uniformly—to x. So our task can be restated as follows: prove that the uniform limit of a sequence of Cauchy sequences is Cauchy. This is somewhat analogous to the fact that the uniform limit of a sequence of continuous functions is continuous (cf. Q.4 of Tutorial 4.)

Let $\varepsilon > 0$. Choose $K \in \mathbb{Z}^+$ such that $d(x^{(k)}, x) < \varepsilon/3$ for all $k \geq K$. Choose $N \in \mathbb{Z}^+$ such that $|x_m^{(K)} - x_n^{(K)}| < \varepsilon/3$ for all $n, m > N$. Then for all $n, m > N$ we have

$$\begin{aligned} |x_m - x_n| &\leq |x_m - x_m^{(K)}| + |x_m^{(K)} - x_n^{(K)}| + |x_n^{(K)} - x_n| \\ &< \sup_{i \in \mathbb{Z}^+} |x_m - x_m^{(K)}| + \tfrac{\varepsilon}{3} + \sup_{i \in \mathbb{Z}^+} |x_i^{(K)} - x_i| \\ &= d(x, x^{(K)}) + \tfrac{\varepsilon}{3} + d(x^{(K)}, x) \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This shows that (x_i) is a Cauchy sequence, as required.

6. Let $X = (0, 1)$ with the Euclidean metric d. Give an example of a nested sequence (A_n) of non-empty closed sets in X with $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$, but $\bigcap_{n=1}^{\infty} A_n = \emptyset$. (The *diameter*, $\operatorname{diam}(A)$, of a subset A of a metric space, is the supremum of the set $\{ d(x, y) \mid x, y \in A \}$, if this set is bounded.)

Solution.

Note that $X = (0, 1)$ is not complete, because it is not closed in $\mathbb{R}$. For example, a sequence in $(0, 1)$ converging in $\mathbb{R}$ to the point 0 will be a Cauchy sequence in $(0, 1)$ with no limit in $(0, 1)$.

Put $A_n = (0, \frac{1}{n}]$. This gives a nested sequence of subsets of X. Each $A_n = [0, 1] \cap X$ is closed in X as $[0, 1]$ is closed $\mathbb{R}$. (Recall that if Y is a subspace of a topological space X then the closed sets of Y are all sets of the form $Y \cap C$, where C is a closed subset of X). Also $d(A_n) = \frac{1}{n} \to 0$ as $n \to \infty$. However $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

7. Let $X = (X, d)$ be a metric space and $\mathrm{CS}(X)$ the collection of all Cauchy sequences in X. For (x_n) and (y_n) in $\mathrm{CS}(X)$, define

$$(x_n) \sim (y_n) \quad \text{if and only if} \quad \lim_{n \to \infty} d(x_n, y_n) = 0.$$

Show that $\sim$ is an equivalence relation on $\mathrm{CS}(X)$.

Solution.

If (x_n) is any Cauchy sequence then $d(x_n, x_n) = 0 \to 0$ as $n \to \infty$. So the relation is reflexive. It is symmetric, since if (x_n) and (y_n) are Cauchy sequences with $(x_n) \sim (y_n)$ then $d(y_n, x_n) = d(x_n, y_n) \to 0$ as $n \to \infty$. Finally, it is symmetric, since if (x_n), (y_n) and (z_n) are Cauchy sequences with $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$ then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, z_n) = 0$, so that by the triangle inequality

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \to 0 + 0 = 0$$

as $n \to \infty$, giving $\lim_{n\to\infty} d(x_n, z_n) = 0$ by the squeeze law.