## Summary of week 4 (lectures 10, 11 and 12)

We have seen that if $F$ is any field then $F^n$, the set of all $n$-tuples of scalars, is a vector space over $F$. The aim of this week's lectures was, roughly speaking, to show that there are essentially no other vector spaces besides these.

Really, of course, there are other vector spaces. But they are, in a natural sense, equivalent to the spaces $F^n$. If some set $S$ has the property that there is a one-to-one correspondence between the elements of $S$ and the elements of $F^n$ then one can use the addition and scalar multiplication operations on $F^n$ and the one-to-one correspondence to define addition and scalar multiplication operations on $S$, and thereby give $S$ the structure of a vector space. But the vector space created by this process is really just a copy of the space $F^n$. The names of the elements have been changed, but the vector space structure is the same. It turns out to be true that every vector space is a copy, in this sense, of some space $F^n$.

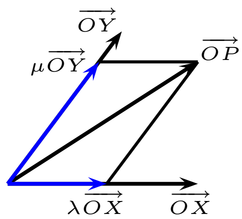
A proviso needs to be inserted at this point. The last sentence of the previous paragraph is not true if $n$ is only allowed to take finite values. But since this course is not primarily about set theory, a discussion of infinite numbers is beyond the scope of the course. So our discussion is restricted to cases in which $n$ is finite.

You are, no doubt, very familiar with the fact that the set

$$\mathbb{R}^2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x,\, y \in \mathbb{R} \}$$

can be identified (via Cartesian coordinates) with the Euclidean plane. We know that $\mathbb{R}^2$ is a vector space over $R$, and so it follows that the Euclidean plane is a vector space over $\mathbb{R}$. However, it should be noted that the Euclidean plane was an object of mathematical interest before cartesian coordinates were invented; furthermore, the addition and scalar multiplication operations that make the plane into a vector space can be defined without reference to a coordinate system. One needs only to choose an origin $O$, to be the zero element in the vector space structure, and then addition and scalar multiplication are defined using concepts of Euclidean geometry. It is customary to identify the vector corresponding to a point $P$ in the plane with the directed line segment $\overrightarrow{OP}$. See Example #2 on p. 53 of [VST].

Now choose any points $X, Y$ in the plane such that $O$, $X$ and $Y$ are not collinear. It is easy to see that for any $P$ in the plane there are scalars $\lambda$, $\mu$ such that $\overrightarrow{OP} = \lambda\overrightarrow{OX} + \mu\overrightarrow{OY}$. The values of $\lambda$ and $\mu$ are found by by drawing lines




through $P$ parallel to $OY$ and $OX$, as shown in the diagram. The points where these lines cut OX and OY correspond to scalar multiples of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ whose sum is $\overrightarrow{OP}$. The scalars $\lambda$ and $\mu$ are called the *coordinates of* $\overrightarrow{OP}$ *relative to the basis* $(\overrightarrow{OX}, \overrightarrow{OY})$.

**Definition.** Let $V$ be a vector space over the field $F$. We say that vectors $v_1, v_2, \ldots, v_d \in V$ form a *basis* of $V$ if they are linearly independent and span $V$.

If, as above, $X$ and $Y$ are points in the plane such that $O$, $X$ and $Y$ are not collinear then $\overrightarrow{OX}$ is not a scalar multiple of $\overrightarrow{OY}$ and $\overrightarrow{OY}$ is not a scalar multiple of $\overrightarrow{OX}$. It follows readily that $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are linearly independent and (as we saw above) span the plane. So they do indeed form a basis in the sense of the above definition.

**Definition.** A vector space $V$ is said to be *finitely generated* or *finitely spanned* if there is some finite sequence $v_1, v_2, \ldots, v_n$ of vectors in $V$ with the property that $V = \mathrm{Span}(v_1, v_2, \ldots, v_n)$.

Our discussion above showed that the Euclidean plane is a finitely generated vector space, having a basis consisting of two vectors. Note that there are many different bases: any choice of $X$ and $Y$ such that $O$, $X$ and $Y$ are not collinear gives a basis. However, every basis for the plane has exactly two elements. More generally, if a vector space $V$ has a basis with $d$ elements then all other bases for $V$ will also have $d$ elements. The number of elements in a basis is called the *dimension* of the space.

Three dimensional Euclidean space is quite analogous to the Euclidean plane. After choosing a point $O$ as the origin, addition and scalar multiplication are defined exactly as for the plane. If $X$, $Y$ and $Z$ are any points such that $O$, $X$, $Y$ and $Z$ are not coplanar, then $\overrightarrow{OX}$, $\overrightarrow{OY}$ and $\overrightarrow{OZ}$ form a basis.

The space $\mathbb{R}^n$, consisting of all $n$-tuples of real numbers, is easily seen to be a finitely generated vector space over $\mathbb{R}$. The $n$ vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

form a basis of $\mathbb{R}^n$. As we shall see later, there are many other bases of $\mathbb{R}^n$ besides this one.

As was mentioned in an earlier lecture, $\mathbb{R}^n$ can be identified with the space of real valued functions on the set $\{1, 2, \ldots, n\}$. Observe that the dimension is $n$. To find an example of a vector space that is not finitely generated we only have to consider the space of scalar valued functions on some infinite set. Thus $\mathscr{F}$, the vector space consisting of all functions from $\mathbb{R}$ to $\mathbb{R}$ is not finitely generated.

For each nonnegative integer $i$ let us define $f_i \in \mathscr{F}$ by $f_i(x) = x^i$ for all $x \in \mathbb{R}$. A *polynomial function* (over $\mathbb{R}$) is a function that can be expressed as a linear combination of $f_i$'s for various $i$. Let us write $\mathscr{P}$ for the set of all polynomial functions over $\mathbb{R}$. That is, $\mathscr{P}$ is the set of all functions $f: \mathbb{R} \to \mathbb{R}$ such that for some integer $n \geq 0$ and some coefficients $a_0, a_1, \ldots, a_n$,

$$f = a_0 f_0 + a_1 f_1 + \cdots + a_n f_n.$$

That is,

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad \text{for all } x \in \mathbb{R}.$$

It is not hard to show that this formula defines the zero function if and only if all the coefficients $a_i$ are zero. If $f$ is a nonzero polynomial then the *degree* of $f$ is by definition the largest $n$ such that the coefficient of $x^n$ is nonzero. We shall adopt the convention that the degree of the zero polynomial is $-\infty$.

Let $d$ be a nonnegative integer, and define $\mathscr{P}_d$ to be the set of all polynomials of degree at most $d$ (including the zero polynomial). Then $\mathscr{P}_d$ is a finitely generated vector space, and it is easily seen that $f_0, f_1, \ldots, f_d$ form a basis of $\mathscr{P}_d$. Thus $\mathscr{P}_d$ has dimension $d+1$. The space $\mathscr{P}$ of all polynomial functions is not finitely generated.

Lectures 10, 11 and 12 included proofs of the following results from [VST]: Lemma 4.2, Corollary 4.3, Lemma 4.4, Proposition 4.13, Theorem 4.14, Proposition 4.6, Proposition 4.7, Proposition 4.8 and Proposition 4.11. Students are strongly urged to read the proofs given in [VST], and to learn all the results listed in §4b.

The following definition (6.1 of [VST]) was stated at the end of Lecture 12:

**Definition.** Let $V$ and $W$ be vector spaces over $F$. A linear function $\phi: V \to W$ that is bijective is called an *isomorphism.* If there is an isomorphism from $V$ to $W$ then we say that $V$ and $W$ are *isomorphic.*

Isomorphic vector spaces can be regarded as copies of each other: for each element of one of the spaces there is a unique corresponding element in the other space, and the addition and scalar multiplication operations are preserved by this correspondence (in the sense that if $v$ corresponds to $w$ and $v'$ to $w'$ then $v+w$ corresponds to $v'+w'$, and $\lambda v$ corresponds to $\lambda v'$ for all scalars $\lambda$).

**Theorem.** *Let $V$ be a vector space over $F$ and $v_1, v_2, \ldots, v_d$ a basis for $V$. Then the function $T: F^d \to V$ defined by*

$$T\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_d\end{pmatrix} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_d v_d$$

*is an isomorphism.*

Informally, this theorem says that $V$ is a copy of $F^d$. The fact that $T$ is a bijective function (that is, a one-to-one correspondence) is Proposition 4.15 of [VST]. The fact that $T$ is linear is part of Theorem of [VST]. We shall have some more to say about these matters in the next one or two lectures.