THE UNIVERSITY OF SYDNEY

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 12

1. Let A be an $n\times n$ matrix whose rank is less than n. Prove that 0 is an eigenvalue of A.

2. Let V be a vector space and S and T subspaces of V such that $V = S \oplus T$. Prove or disprove the following assertion:

If U is any subspace of V then $U = (U \cap S) \oplus (U \cap T)$.

3. (*i*) Let A, B and C be $n \times n$ matrices, and suppose that the column space of B equals the column space of C. Prove that the column space of AB equals that of AC. (Hint: Use Proposition 7.16 of the text.)

(*ii*) Let A be an $n \times n$ matrix and suppose that the rank of A^4 is the same as the rank of A^3. Prove that A^5 and all higher powers of A also have this same rank. (Hint: Apply Part (*i*) with $B = A^3$ and $C = A^4$.)

4. Let V and W be vector spaces over the field F and let $\boldsymbol{b} = (v_1, v_2, \dots, v_n)$ and $\boldsymbol{c} = (w_1, w_2, \dots, w_m)$ be bases of V and W respectively. Let $\mathrm{L}(V, W)$ be the set of all linear transformations from V to W, and let $\mathrm{Mat}(m\times n, F)$ be the set of all $m \times n$ matrices over F. We know that $\mathrm{Mat}(m \times n, F)$ is a vector space over F, and we have seen in Question 3 of Tutorial 5 that $\mathrm{L}(V, W)$ is too. Let $\Omega\colon \mathrm{L}(V, W) \to \mathrm{Mat}(m \times n, F)$ be the function defined by $\Omega(\theta) = \mathrm{M}_{\boldsymbol{cb}}(\theta)$ for all $\theta \in \mathrm{L}(V, W)$.

(*i*) Prove that Ω is a linear transformation. (Hint: The task is to prove that $\mathrm{M}_{\boldsymbol{cb}}(\phi + \theta) = \mathrm{M}_{\boldsymbol{cb}}(\phi) + \mathrm{M}_{\boldsymbol{cb}}(\theta)$ and $\mathrm{M}_{\boldsymbol{cb}}(\lambda\phi) = \lambda\, \mathrm{M}_{\boldsymbol{cb}}(\phi)$. Now the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta)$ is $\mathrm{cv}_{\boldsymbol{c}}((\phi+\theta)(v_j))$ while the j^{th} columns of $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ and $\mathrm{M}_{\boldsymbol{cb}}(\theta)$ are $\mathrm{cv}_{\boldsymbol{c}}(\phi(v_j))$ and $\mathrm{cv}_{\boldsymbol{c}}(\theta(v_j))$. Use the definition of $\phi+\theta$ and fact that $x \mapsto \mathrm{cv}_{\boldsymbol{c}}(x)$ is linear to prove that the j^{th} column of $\mathrm{M}_{\boldsymbol{cb}}(\phi+\theta)$ is the sum of the j^{th} columns of $\mathrm{M}_{\boldsymbol{cb}}(\phi)$ and $\mathrm{M}_{\boldsymbol{cb}}(\theta)$.)

(*ii*) Prove that the kernel of Ω is $\{z\}$, where $z\colon V \to W$ is the zero function.

(*iii*) Prove that Ω is a vector space isomorphism. (Hint: By the first two parts we know that Ω is linear and injective; so surjectivity is all that remains. That is, given a $m \times n$ matrix M we must show that there is a linear transformation θ from V to W having M as its matrix. Now the coefficients of M determine what $\theta(v_i)$ has to be for each i, and Theorem 4.18 guarantees that such a linear transformation exists.)

(*iv*) Find a basis for $\mathrm{L}(V, W)$. (Hint: (Find a basis of $\mathrm{Mat}(m \times n, F)$ first. The corresponding linear transformations will give the desired basis of $\mathrm{L}(V, W)$.)