The University of Sydney

MATH2902 Vector Spaces

(http://www.maths.usyd.edu.au/u/UG/IM/MATH2902/)

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Tutorial 1

Let X and Y be arbitrary nonempty sets, and $f: X \to Y$ a function. A function $g: Y \to X$ is a *right inverse* of f if the composite function fg is the identity on Y. Similarly g is a *left inverse* of f if gf is the identity on X.

1. Let A be a set with 5 elements and B a set with 4 elements. Let the elements of A be called a_1, a_2, a_3, a_4 and a_5, so that $A = \{a_1, a_2, a_3, a_4, a_5\}$. Similarly let $B = \{b_1, b_2, b_3, b_4\}$.

(*i*) Describe three different surjective functions with domain A and codomain B, and three different injective functions with domain B and codomain A.

(*ii*) Find right inverses for each of the three surjective functions you found in (*i*), and left inverses for each of the injective functions.

Solution.

(*i*) For instance

$$f_1\colon \begin{array}{l} a_1 \mapsto b_1 \\ a_2 \mapsto b_2 \\ a_3 \mapsto b_3 \\ a_4 \mapsto b_4 \\ a_5 \mapsto b_1 \end{array} \qquad f_2\colon \begin{array}{l} a_1 \mapsto b_1 \\ a_2 \mapsto b_2 \\ a_3 \mapsto b_3 \\ a_4 \mapsto b_4 \\ a_5 \mapsto b_2 \end{array} \qquad f_3\colon \begin{array}{l} a_1 \mapsto b_1 \\ a_2 \mapsto b_2 \\ a_3 \mapsto b_3 \\ a_4 \mapsto b_4 \\ a_5 \mapsto b_4 \end{array}$$

are three surjective functions from A to B, and

$$g_1\colon \begin{array}{l} b_1 \mapsto a_1 \\ b_2 \mapsto a_2 \\ b_3 \mapsto a_3 \\ b_4 \mapsto a_4 \end{array} \qquad g_2\colon \begin{array}{l} b_1 \mapsto a_1 \\ b_2 \mapsto a_2 \\ b_3 \mapsto a_3 \\ b_4 \mapsto a_5 \end{array} \qquad g_3\colon \begin{array}{l} b_1 \mapsto a_1 \\ b_2 \mapsto a_2 \\ b_3 \mapsto a_4 \\ b_4 \mapsto a_5 \end{array}$$

are three injective functions from B to A.

(*ii*) If $i = 1$, 2, 3 or 4 then $(f_1 g_1)(b_i) = f_1\big(g_1(a_i)\big) = f_1(a_i) = b_i$, showing that f_1 is a left inverse of g_1, and g_1 a right inverse of f_1. Moreover,

since $f_1(a_i) = f_2(a_i) = f_3(a_i)$ for $i \le 4$, exactly the same calculations show that g_1 is also a right inverse of both f_2 and f_3 as well. Similarly, for $1 \le i \le 3$ we find that $(f_3 g_2)(b_i) = f_3(a_i) = b_i$, while $(f_3 g_2)(b_4) = f_3(a_5) = b_4$, so that $f_3 g_2$ is the identity on B. Thus f_3 is a left inverse of g_2. Finally, one can check that the function defined by $a_1 \mapsto b_1$, $a_2 \mapsto b_2$, $a_3 \mapsto b_1$, $a_4 \mapsto b_3$ and $a_5 \mapsto b_4$ is a left inverse for g_3.

Note that these examples show that it is sometimes possible for a function to have several left or right inverses.

2. Let A and B be arbitrary nonempty sets.

(*i*) Let $f: A \to B$ be an arbitrary function. Prove that if f has a right inverse then f must necessarily be surjective, and prove that if f has a left inverse then f is necessarily injective.

(*ii*) Prove that if f is surjective then it has a right inverse. Prove also that if f is injective then it has a left inverse.

(*iii*) Prove that if f has both a right inverse and a left inverse then they are equal.

Solution.

(*i*) Assume that $g: B \to A$ is a left inverse of f. Suppose that $x, y \in A$ satisfy $f(x) = f(y)$. Since $gf = \iota_A$, the identity on A, we have

$$x = \iota_A(x) = (gf)(x) = g(f(x)) = g(f(y)) = (gf)(y) = \iota_A(y) = y.$$

So $x = y$ whenever $f(x) = f(y)$; that is, f is injective.

Suppose that $g: B \to A$ is a right inverse of f, and let $b \in B$. Then

$$b = \iota_B(b) = (fg)(b) = f(a)$$

where $a = g(b)$. So for each $b \in B$ there is an $a \in A$ with $f(a) = b$. Thus f is surjective.

(*ii*) Assume that f is injective. Then for each $b \in B$ there is at most one $a \in A$ with $f(a) = b$. Define a function $g: B \to A$ as follows. If $b \in B$ and $b = f(a)$ for some $a \in A$ define $g(b) = a$. If there is no $a \in A$ with $b = f(a)$ it is irrelevant how $g(b)$ is defined; for instance, we may pick some fixed $a_0 \in A$ (since A is nonempty) and define $g(b) = a_0$ for all such b. Now for all $a \in A$ we have $(gf)(a) = g(b)$ where $b = f(a)$, and the definition of g gives $g(b) = a$ (since there is no other element of A mapped to b by f). Thus gf is the identity, and g is a left inverse of f.

Assume that f is surjective, and define $g: B \to A$ as follows. For each $b \in B$ there is at least one $a \in A$ with $f(a) = b$; we choose any such a and define $g(b) = a$. (The particular choices that are made for each b

are irrelevant, and so there may be many suitable functions g.) Then for each $b \in B$ we have that $g(b)$ satisfies $f(g(b)) = b$. That is, $(fg)(b) = b$ for all b, and fg is the identity. So g is a right inverse of f.

(*iii*) Assume that h is a left inverse of f and k is a right inverse of f. Then we have $h(f(a)) = a$ for all $a \in A$ and $f(k(b)) = b$ for all $b \in B$. Let $b \in B$ and write $a = k(b)$. Then

$$h(b) = h(f(k(b)) = h(f(a)) = a.$$

Thus $h(b) = k(b)$ for all $b \in B$; that is, $h = k$.

3. If f and g are functions with domain X and codomain Y then the correct way to prove that $f = g$ is to prove that $f(x) = g(x)$ for all $x \in X$. Similarly, if A and B are $m \times n$ matrices then proving that $A = B$ is done by proving that $A_{ij} = B_{ij}$ for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$.

Prove that if A is an $m \times n$ matrix and I is the $n \times n$ identity matrix then $AI = A$. Prove also that if J is the $m \times m$ identity then $JA = A$.

Solution.

Let $(i, j) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$; We must prove that $(AI)_{ij} = A_{ij}$. By definition of matrix multiplication we have

$$(AI)_{ij} = \sum_{k=1}^{n} A_{ik}\delta_{kj}$$

and since δ_{kj} is zero unless $k = j$ all terms in the sum corresponding to other values of k vanish. Thus $(AI)_{ij} = A_{ij}\delta_{jj} = A_{ij}$, as required.

Similarly, $(JA)_{ij} = \sum_{k=1}^{m} \delta_{ik}A_{kj} = \delta_{ii}A_{ij} = A_{ij}$, and since this holds for all values of i and j it follows that $JA = A$.

4. Let A be an $n \times n$ matrix. A matrix B is an *inverse* of A if $AB = BA = I$. Use the previous exercise and associativity of matrix multiplication to prove that if B and C are both inverses of A then $B = C$.

Solution.

$B = BI = B(AC) = (BA)C = IC = C.$

5. Let F be any field. Prove that if x, $y \in F$ are such that $xy = 0$ then either $x = 0$ or $y = 0$.

Solution.

Let us first prove that $z0 = 0$, for all $z \in F$. By field axiom (ii) we have

$$\begin{aligned} 0 + z0 &= z0 \\ &= z(0 + 0) \qquad \text{(axiom (ii) again)} \\ &= z0 + z0 \qquad \text{(by axiom (ix)).} \end{aligned}$$

By axiom (iii) there is an element $b \in F$ such that $z0 + b = 0$, and adding this to both sides of the equation just proved gives, with a few applications of axioms (ii) and (i),

$$0 = 0{+}0 = 0{+}(z0{+}b) = (0{+}z0){+}b = (z0{+}z0){+}b = z0{+}(z0{+}b) = z0{+}0 = z0.$$

Now let x, $y \in F$ and assume that $xy = 0$. Assume $x \neq 0$. Then by field axiom (vii) there exists $z \in F$ with $zx = 1$. This gives

$$y = 1y = (zx)y = z(xy) = z0 = 0.$$

We have now shown that if $x \neq 0$ then $y = 0$; that is, either $x = 0$ or $y = 0$.