

### Computer Tutorial 2

The purpose of this tutorial is to introduce subspaces of vector spaces, to demonstrate how MAGMA can be used to solve matrix equations and how to use matrices to calculate projections and lines of best fit.

The MAGMA commands you will learn in this tutorial are:

sub	KMatrixSpace	Dimension	Transpose	Solution
-----	--------------	-----------	-----------	----------

1. In MAGMA, create the subspace of  $(\mathbb{R}^5)'$  spanned by the vectors  $(1,1,0,1,1)$ ,  $(0,1,0,1,-1)$  and  $(3,1,0,1,5)$ . Do this by carrying out the following steps:

- Define the real field  $R := \text{RealField}()$ ;
- Create a vector space  $V$  of dimension 5 over  $R$  (as in Computer Tutorial 1).
- Declare the vectors above as  $a1$ ,  $a2$  and  $a3$ . For example,
 

```
a1 := V![1,1,0,1,1];
```
- Create the subspace  $W$  spanned by  $a1$ ,  $a2$  and  $a3$  using the command
 

```
W := sub< V | a1, a2, a3 >;
```
- Find the dimension of  $W$ . (Magma has a function `Dimension`.)
- Are the vectors  $a1$ ,  $a2$  and  $a3$  linearly independent?

*Solution.*

<pre>&gt; R := RealField();</pre>	<pre>&gt; a3 := V![3,1,0,1,5];</pre>
<pre>&gt; V := VectorSpace(R,5);</pre>	<pre>&gt; W := sub&lt; V   a1, a2, a3 &gt;;</pre>
<pre>&gt; a1 := V![1,1,0,1,1];</pre>	<pre>&gt; Dimension(W);</pre>
<pre>&gt; a2 := V![0,1,0,1,-1];</pre>	<pre>2</pre>

Since they span a two-dimensional space rather than a three-dimensional space,  $a1$ ,  $a2$  and  $a3$  linearly dependent.

2. Consider the following vectors in  $\mathbb{R}^5$ :

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} -1 \\ 2 \\ -3 \\ 4 \\ -5 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 5 \\ -4 \\ 3 \\ 6 \end{pmatrix}.$$

We wish to calculate the projection of  $v$  onto the subspace  $W$  of  $\mathbb{R}^5$  with basis  $\{b_1, b_2, b_3\}$ . Since MAGMA uses row vectors rather than column vectors, we have to transpose everything.

- We have already (in Exercise 1) created the field  $R$  and  $V$ , a 5-dimensional space over  $R$ . Define  $b1 := V![1,1,0,1,1]$ ; and similarly define  $b2$ ,  $b3$  and  $v$  to be the transposes of  $b_2$ ,  $b_3$  and  $v$  above.

- Create the vector space of  $3 \times 5$  matrices using the command

```
M := KMatrixSpace(R,3,5);
```

- Create the matrix  $B$  with rows  $b1$ ,  $b2$  and  $b3$  using the command

```
B := M![b1,b2,b3];
```

(This is the transpose of the matrix  $A$  whose columns are  $b_1$ ,  $b_2$  and  $b_3$ .)

- Print  $B$  and  $v$  to check that you have entered everything correctly, and then define  $A := \text{Transpose}(B)$ ;

- According to the theory developed in lectures, the projection of  $v$  onto  $W$  is the vector  $Ax$ , where  $A^T Ax = A^T v$ . Taking transposes this equation becomes  $x^T A^T A = v^T A$ , since transposing reverses multiplication. We can obtain the vector  $x^T$  via the command

```
x := Solution(B*A,v*A);
```

(Explanation: if  $M$  is a matrix and  $b$  a (row) vector then `Solution(M,b)` is a vector  $x$  that is a solution of the matrix equation  $xM = b$ .)

- The projection is  $p = Ax$ . Transposing (remembering that  $B = A^T$  and that the vector  $x$  that MAGMA has found is actually  $x^T$ ), we see that

```
x * B;
```

will print out the row vector that is the transpose of  $p$ .

*Solution.*

<pre>&gt; b1 := V![1,1,0,1,1];</pre>	<pre>(1 5 -4 3 6)</pre>
<pre>&gt; b2 := V![0,1,0,1,0];</pre>	<pre>&gt; A := Transpose(B);</pre>
<pre>&gt; b3 := V![-1,2,-3,4,-5];</pre>	<pre>&gt; x := Solution( B*A,v*A );</pre>
<pre>&gt; M := KMatrixSpace(R,3,5);</pre>	<pre>&gt; x;</pre>
<pre>&gt; B := M![b1,b2,b3];</pre>	<pre>(7/2 1/2 0)</pre>
<pre>&gt; v := V![1,5,-4,3,6];</pre>	<pre>&gt; x * B;</pre>
<pre>&gt; B,v;</pre>	<pre>(7/2 4 0 4 7/2)</pre>
<pre>[1 1 0 1 1]</pre>	
<pre>[0 1 0 1 0]</pre>	
<pre>[-1 2 -3 4 -5]</pre>	

3. Repeat the previous exercise to find the projection of  $(1,0,1,0)^T$  onto the subspace of  $\mathbb{R}^4$  spanned by  $(-2,1,0,1)^T$  and  $(-3,0,1,0)^T$ .

*Solution.*

<pre>&gt; V := VectorSpace(R,4);</pre>	<pre>&gt; B := M![b1,b2];</pre>
<pre>&gt; b1 := V![-2,1,0,1];</pre>	<pre>&gt; x := Solution( B*A,v*A )</pre>
<pre>&gt; b2 := V![-3,0,1,0];</pre>	<pre>where A is Transpose(B);</pre>
<pre>&gt; v := V![1,0,1,0];</pre>	<pre>&gt; x * B;</pre>
<pre>&gt; M := KMatrixSpace(R,2,4);</pre>	<pre>(2/3 -1/3 0 -1/3)</pre>

4. Find the least squares line of best fit for the following data.

atomic no.	$\log_{10}$ of half-life (seconds)
104	1.8
105	1.5
106	-0.05
107	-0.97
108	-2.7
109	-2.5

There are six data points  $(x_1, y_1), \dots, (x_6, y_6)$ . By the theory given in lectures the line of best fit is  $y = a + bx$ , where  $a(1, 1, 1, 1, 1, 1) + b(x_1, x_2, x_3, x_4, x_5, x_6)$  is as close as possible to  $(y_1, y_2, y_3, y_4, y_5, y_6)$ . Use the following commands:

```
R := RealField(10);
V := VectorSpace(R,6);
B := M![1,1,1,1,1,1,104,105,106,107,108,109] where
M is KMatrixSpace(R,2,6);
y := V![1.8,1.5,-0.05,-0.97,-2.7,-2.5];
Solution(B*A,y*A) where A is Transpose(B);
```

Now make a prediction for the half-life of element number 110.

*Solution.*

```
> V := VectorSpace(R,6);
> M := KMatrixSpace(R,2,6);
> B := M![1,1,1,1,1,1,104,
105,106,107,108,109];
> y := V![1.8,1.5,-0.05,
-2.7,-2.5];
> x := Solution( B*A,y*A )
where A is Transpose(B);
> x;
(106.07403517 -1.00056997)
> U := VectorSpace(R,2);
```

Recall that the value of  $x$  gives the coefficients  $\alpha$  and  $\beta$  in the equation  $y = \alpha + \beta x$  of the line of best fit. So in this case the line is  $y = 106.074 - x$ . Evaluating the right hand-side at  $x = 110$  amounts to calculating the dot product of  $(106.074, -1)$  and  $(1, 110)$ , and this explains the three MAGMA commands above following the calculation of  $x$ . Note that  $-3.988\dots$  is the log of the half-life; to get the half-life itself we must raise 10 to this power.

If the prediction is correct, the half-life is about one ten-thousandth of a second.

5. The data below gives the concentration of atmospheric  $\text{CO}_2$  (in ppm) in various years. Find the parabola of best fit.

Year	Concentration
1852	288
1914	300
1932	306
1950	311
1960	317.9
1970	325.6
1980	338.7
1990	354.2

Make a prediction for the year 2040.

If a change of 100 ppm corresponds to a change of  $1^\circ\text{C}$  in the average global temperature, how much warmer will it be then? (Note: In the last ice age the mean global temperature was about  $1^\circ\text{C}$  less than it is now.)

(The last two questions are derived from articles that appeared in “Scientific American”, April 1989.)

*Solution.*

```
> V := VectorSpace(R,8);
> M := KMatrixSpace(R,3,8);
> B := M![1,1,1,1,1,1,1,1852,1914,1932,1950,1960,1970,1980,
1990,1852^2,1914^2, 1932^2,1950^2,1960^2,1970^2,1980^2,1990^2];
> y := V![288,300,306,311,317.9,325.6,338.7,354.2];
> x := Solution( B*A,y*A ) where A is Transpose(B);
> x;
(16070.0948 -16.85201478 0.004498490307)
> P<t>:=PolynomialAlgebra(R);
> f:=P![x[i]: i in [1,2,3]];
> f;
0.004498490307*t^2 - 16.85201478*t + 16070.0948
> Evaluate(f,2040);
412.9019
```

A change of 59 ppm as compared with 2000 means—if this theory is to be believed—that in 2040 things will be about about a half a degree warmer than in 2000.