Hidden Symmetries of the Root Systems

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Symmetry

is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

H.Weyl (Symmetry)

The root systems

Introduced and classified by Killing (1889) (with a correction concerning $\mathbb{F}_4$ by Cartan)

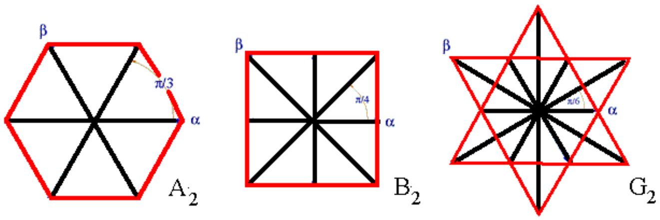
Melting pot for **elementary** mathematics (elementary geometry) and **sophisticated** mathematics (Lie groups, Lie algebras, etc)

Here a first vague description: A **root system** is a finite set Φ of non-zero vectors in Euclidean n-space $\mathbb{R}^n$ which is closed under the reflections induced by its elements, and such that some additional integrality condition (the "crystallographic condition") is satisfied.

The dimension n of the ambient space $\mathbb{R}^m$ is called the **rank** of Φ.

Elementary geometry: The irreducible root systems of rank 2

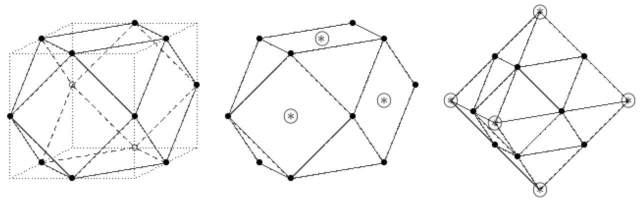
- The regular 6-gone
- The square with the centers of the edges
- The star of David



They are labeled $\mathbb{A}_2, \mathbb{B}_2, \mathbb{G}_2$, respectively.

Elementary geometry: The irreducible root systems of rank 3

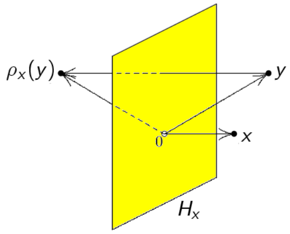
- The **cube-octaedron:** the centers of the edges of a cube.
- The cube-octahedron with the centers of the square faces, thus the centers of the edges and of the faces of a **cube.**
- The **octahedron:** its vertices and the centers of the edges



They are labeled $\mathbb{A}_3, \mathbb{B}_3, \mathbb{C}_3$, respectively.

Closure under internal symmetries:

We take a finite set Φ of non-zero vectors in Euclidean n-space $\mathbb{R}^n$. Any such vector x defines a reflection ρ_x (with reflection hyperplane the hyperplane H_x orthogonal to x). We want that Φ is closed under the reflections ρ_x (thus, if $x, y \in \Phi$, also $\rho_x(y) \in \Phi$)



Even more: we want the following integrality condition: Clearly $\rho_x(y) - y$ is a multiple of x. It should be an **integral** multiple (this is called the "cristallographic" condition).

Consequences:

- Only few angles between the elements of Φ are possible.
- The quotient between the lengths of roots is strongly restricted.

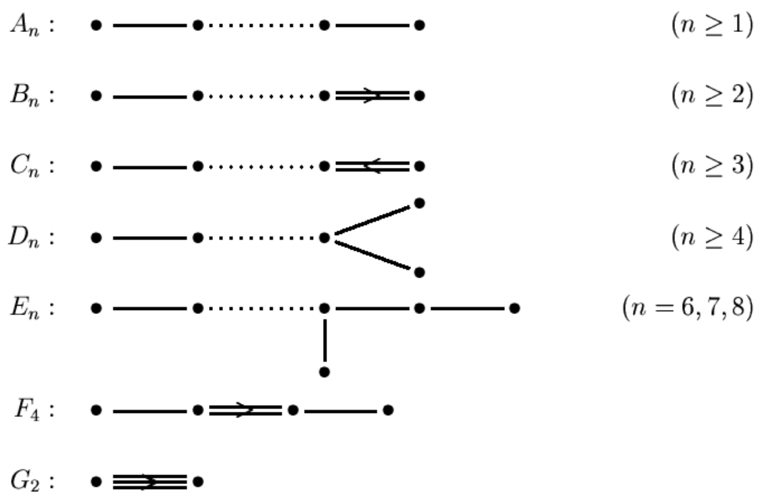
If we deal with an "irreducible" root system in n-space, then:

- The number of roots is divisible by n (to be precise: $|\Phi| = nh$, where h is the "Coxeter number")
- If $n = 3$, the only possible angles are multiples of $\pi/4$ and $\pi/3$, and the quotient between the lengths of roots is 1 or $\sqrt{2}$.
  (As we see, the case $\mathbb{G}_2$ is very special and has no analog in higher dimensions: here we deal with the angle $\pi/6$ and with $\sqrt{3}$ as a possible ratio between the length of roots.)

The root systems can be classified: There are the (Dynkin) types

$$\Delta = \mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \mathbb{F}_4, \mathbb{G}_2$$

Classification of the irreducible root systems:



The corresponding geometries:

A_n	affine geometry	n+1
B_n	orthogonal geometry (odd dim)	2n+1
C_n	symplectic geometry	2n
D_n	orthogonal geometry (even dim)	2n
E_6	Exceptional geometries	27 lines on a cubic surface
E_7		28 double tangents to a quartic curve
E_8		
F_4		
G_2		

Typical applications:

Classification of suitable

- Lie algebras
  Lie groups
  Algebraic groups
- Hyperplane arrangements
- Singularities
- Finite dimensional (associative) algebras:
  hereditary algebras (quivers and species)
  selfinjective algebras
- Cluster algebras
- ...

Let Φ be the root system of type Δ. Recall that this implies:

Φ has many symmetries,

namely for every root x, the reflection ρ_x permutes the roots. The group $W = W(\Delta)$ generated by these reflections ρ_x is called the **Weyl group.** The Weyl group has nice properties, in particular: it is a Coxeter group.

Many properties of the Weyl group (and of Φ) can be formulated in terms of the so-called exponents $e_1, \ldots, e_n$, these are n natural numbers. For example one has:

$$|W| = \prod_i (e_i + 1), \quad |\Phi| = 2\sum_i e_i = nh,$$

where h is called the Coxeter number.

How does one find the Dynkin type? One needs a root basis.

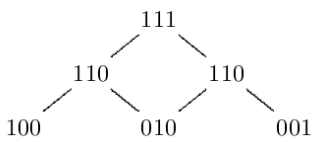
Theorem: *Let Φ be a root system in $\mathbb{R}^n$. There exist roots which form a basis of $\mathbb{R}^n$ such that any root x is either positive or negative.*

(This means: when x is expressed in this basis,
then the coefficients are either all non-negative or non-positive).

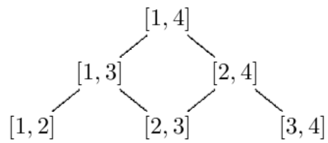
Such a basis is called a **root basis.** its elements the **simple** roots.

The Weyl group acts simply transitive on the set of root bases, and the Dynkin diagram encodes the angles between the basis elements as well as their relative lengths.

$\Phi = \Phi(\Delta)$ the root system of type Δ with a fixed root basis.
$\Phi_+ = \Phi_+(\Delta)$ the set of positive roots.
Φ_+ is a poset via the "nesting" relation:
    $x = y$ iff $y - x$ is a sum of elements of the root basis.
Example: The poset $\Phi_+(\mathbb{A}_n)$ is the inclusion poset of the intervals $[i, j]$, where $1 = j = n$.



The root poset Φ_+ for $\mathbb{A}_3$

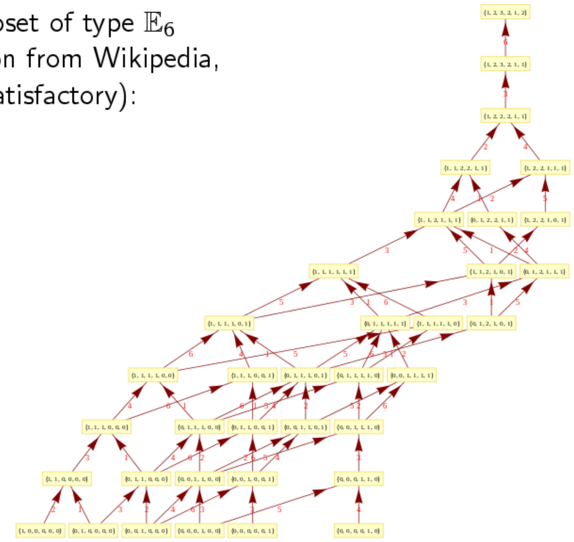


The inclusion poset of the intervals [i,j] with integers $1 \le i < j \le 4$

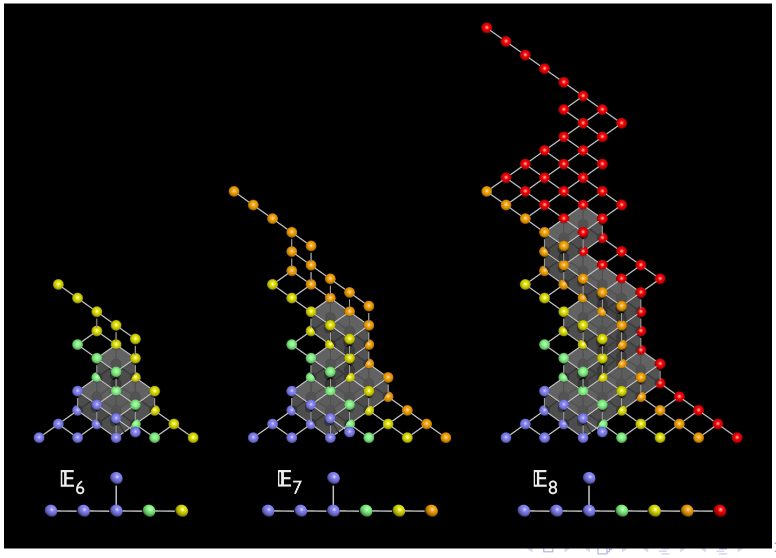
Φ_+ is the object which we want to look at.

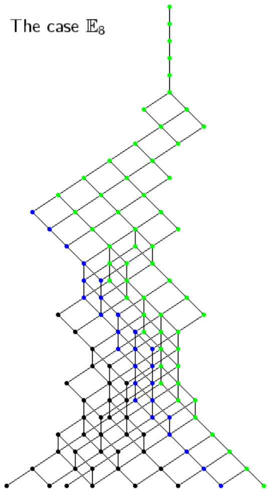
If $x \in \Phi_+$ is the sum of t simple roots, t is called its **height.**

The root poset of type $\mathbb{E}_6$ (visualization from Wikipedia, not really satisfactory):

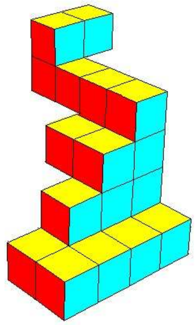


Better: $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$



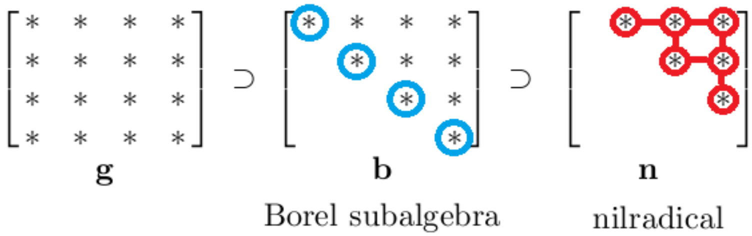


For Δ of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$, the cuboids in Φ_+ correspond bijectively to the non-thin positive roots (Beineke 2013).



Always, all intervalls of Φ_+ are distributive lattices.

Consider the Lie algebra **gl**(4) (the set of (4×4)-matrices with respect to addition and commutation) and its subalgebras **b** and **n**



Marked in red: the weight space decomposition of **n** with respect to the diagonal matrices (Cartan sublgebra, marked in blue).
We see nicely the root poset Φ_+ of type $\mathbb{A}_3$ (slightly tilted):
here it describes positions in a matrix.

Representations of algebras.

Λ a fin dim k-algebra (assoc, with 1, connected, k a field).
Any (fin dim) Λ-module is the direct sum of indecomposable modules and the Krull-Remak-Schmidt theorem asserts: such a decomposition is unique (up to isomorphisms).

A hereditary algebra Λ is called a **Dynkin algebra** provided the set ind Λ of (iso-classes of) indecomposable Λ-modules is finite.

Theorem (Gabriel 1972, Dlab-Ringel 1973): *The indecomposable modules for a Dynkin algebra Λ correspond bijectively to the positive roots of a root system $\Phi(\Delta)$.*

Dlab-Ringel 1979: *If Λ is a Dynkin algebra and $|k| \geq 3$, then the posets* (ind Λ, subfactor-relation) *is isomorphic to* $(\Phi_+(\Delta), \leq)$.
(A module X is a subfactor of the module Y iff there is a chain of submodules $Y'' \subseteq Y' \subseteq Y$ with Y'/Y'' isomorphic to X).

	Positive roots Φ_+	**Complete root system Φ**
Groups	Borel subgroup	Simple group
Algebras	module category	Derived category

Until quite recently, for most geometers and algebraists the set Φ_+ was just a half of Φ, it was not at all considered as a mathematical object in its own right.

The aim of the lecture is to point out the relevance of Φ_+ itself.

Let us stress: By definition, the root system Φ has a lot of symmetries — the reflections ρ_x for all the roots x.
All these symmetries are destroyed when we look at Φ_+.

But we will see: The poset $\Phi_+ = (\Phi_+, \leq)$ itself as well as sets derived from this poset **have several unexpected symmetries.**

We are going to present a survey about some of these considerations (as we will see, they are due to mathematicians working in very different areas).

We will start with an old, but less known result.

Hidden symmetries I:

The number of roots of fixed height.

The number $r(t)$ of roots of height t.

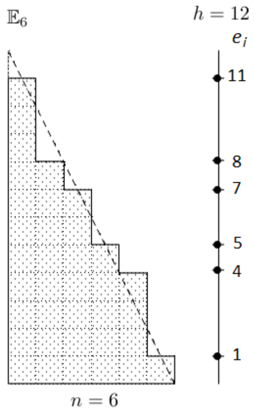
Φ_t the roots of height t.
$r(t) = |\Phi_t|$.

Theorem (Kostant 1959). *The sequence* $(r(1), r(2), \ldots)$ *is a partition, its transpose is the partition* $(e_n, e_{n-1}, \ldots, e_1)$ *of exponents.*

For example, for $\mathbb{E}_6$:

$$(r(1), r(2), \ldots) = (6, 5, 5, 5, 4, 3, 3, 2, 1, 1, 1),$$

$$(e_6, e_5, \ldots, e_1) = (11, 8, 7, 5, 4, 1).$$



This means: *The root poset Φ_+ determines the exponents of W, thus also the degrees of W.*

Exponents and degrees are defined for any finite Coxeter group.
We recall the definitions.

Take a Coxeter element c in W (note that all Coxeter elements are conjugate) and consider its eigenvalues, say $\zeta^{e_1}, \dots, \zeta^{e_n}$ where $\zeta \in \mathbb{C}$ is a primitive hth root of unity and $e_1 \le \cdots \le e_n$ are natural numbers. The numbers e_i are called the **exponents** of W.

The operation of W on $\mathbb{R}^n$ yields an operation of W on the ring of regular functions on $\mathbb{R}^n$, this is the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$. The ring of W-invariants is generated by n homogeneous polynomials, say of degree $d_1 \le \cdots \le d_n$. The d_i are called the **degrees** of W and there is the relation:

$$d_i = e_i + 1, \quad \text{for all } 1 \le i \le n.$$

W operates on Φ, but not on Φ_+.
Still, important properties of W can be read off from Φ_+.

The number $r(t)$ of roots of height t.

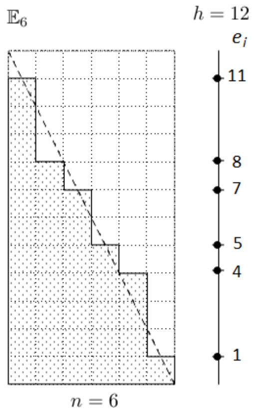
$$r(i) + r(h-i+1) = n, \quad \text{for } 1 \le i \le h.$$

Proof: this follows directly from the symmetry

$$e_i = e_{n+1-i}.$$

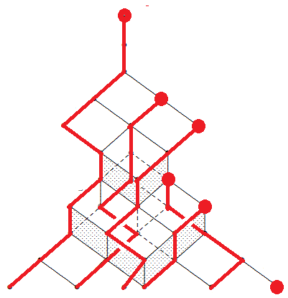
Consequently: $r(1) = n$ implies that $r(h) = 0$, $r(2) = n - 1$ implies that $r(h-1) = 1$.

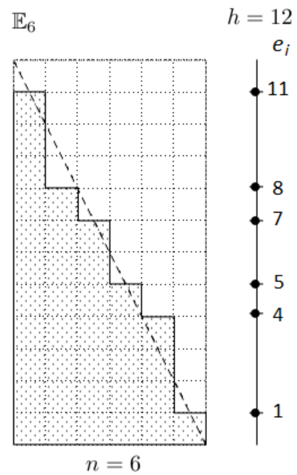
Thus there is a unique maximal element in Φ_+, the unique root with height $h - 1$.



Even: Φ_+ *can be written as the disjoint union of solid subchains* C_i (with C_i of length e_i)

("solid subchain" = neighbors in the subchain are neighbors in the poset)





Note that these assertions refine the equalities $|\Phi| = 2\sum_i e_i = nh$.

Hidden symmetries II:

The number of antichains in Φ_+ of fixed cardinality.

Antichains in a root poset.

(an antichain in a poset is a set of pairwise incomparable elements)

$A(\Phi_+)$ the set of all antichains in the root poset $\Phi_+ = \Phi_+(\Delta)$.

This is a lattice: the lattice $NN(\Delta)$ of "non-nesting partitions" (recall: the partial order of Φ_+ is called the "nesting" relation), it is isomorphic to the lattice $NC(\Delta)$ of "non-crossing partitions" of type Δ and

$$\boxed{|A(\Phi_+)| = \Delta\text{-Catalan number} = |W|^{-1}\prod_i(h + e_i + 1).}$$

$\mathbb{A}_{n-1}$	$\mathbb{B}_n, \mathbb{C}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$	$\mathbb{F}_4$	$\mathbb{G}_2$
$\frac{1}{n+1}\binom{2n}{n}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8

$A_t(\Phi_+)$ the set of t-antichains (the antichains of cardinality t).

There is one n-antichain: the root basis,
and one 0-antichain: the empty set.

Theorem:

$$|A_t(\Phi_+)| = |A_{n-t}(\Phi_+)| \quad \text{for } 1 \le t \le n.$$

Thus, the numbers are palindromic
(in fact, the lattice $A(\Phi_+)$ is self-dual).

In particular: $|A_{n-1}(\Phi_+)| = |\Phi_+|$.

Note: Φ_1 is an n-antichain, Φ_t is an $(n-1)$-antichain for $2 \le t \le e_2$.

If x belongs to an $(n-1)$-antichain, then either $x \in \Phi_{\le e_2}$ or else $\Delta = \mathbb{E}_6$ and x is the non-thin root of height 5.

$|A_q(\Phi_+(\Delta))| = N(\Delta, q)$ the Δ-Narayana numbers.

$$N(\mathbb{A}_n, q) = \sum_{t=0}^{n} \frac{1}{n+1}\binom{n+1}{t}\binom{n+1}{t+1} q^t$$

$$N(\mathbb{B}_n, q) = \sum_{t=0}^{n} \binom{n}{t}^2 q^t$$

$$N(\mathbb{D}_n, q) = 1 + \sum_{t=1}^{n-1} \left[\binom{n}{t}^2 - \frac{n}{n-1}\binom{n-1}{t-1}\binom{n-1}{t}\right] q^t + q^n$$

$$N(\mathbb{E}_6, q) = 1 + 36q + 204q^2 + 351q^3 + 204q^4 + 36q^5 + q^6$$

$$N(\mathbb{E}_7, q) = 1 + 63q + 546q^2 + 1470q^3 + 1470q^4 + 546q^5 + 63q^6 + q^7$$

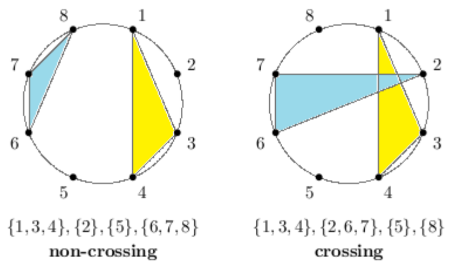
$$N(\mathbb{E}_8, q) = 1 + 120q + 1540q^2 + 6120q^3 + 9518q^4 + 6120q^5 + 1540q^6 + 120q^7 + q^8$$

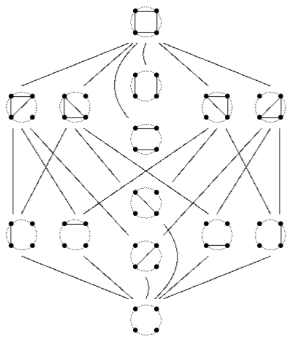
$$N(\mathbb{F}_4, q) = 1 + 24q + 55q^2 + 24q^3 + q^4$$

$$N(\mathbb{G}_2, q) = 1 + 6q + q^2.$$

The lattice $NC(\mathbb{A}_n)$ of non-crossing partitions of $\{1, ..., n+1\}$

For $n = 7$:





The case $\mathbb{A}_3$

Studied systematically by Kreweras (1972), also Becker, Motzkin (1948).

It plays a major role in **free probability theory** (Voiculescu 1985, Speicher 1994, Biane 1997).

The lattice $NC(W)$ of generalized non-crossing partitions for any finite Coxeter group W.

Brady, Brady-Watt, Bessis (2002/3)
Reiner (1997), Athanasiadis (2000)
Armstrong (Memoirs AMS 2009),
and many others (Garsia, Haiman, Panyushev, ...)

The definition of $NC(W) = NC(W, c)$ uses the absolute order $\leq_a$ on W and the **choice of a Coxeter element** c, thus for $W = W(\Delta)$ the **choice of an orientation** of Δ. Namely:

If $w \in W$, let $|w|_a = \min\{t \mid w \text{ is a product of } t \text{ reflections}\}$. Let $v \leq_a w$ iff $|v|_a + |v^{-1}w|_a = |w|_a$ (this is called **absolute order**).

$$NC(W) = \{w \in W \mid w \leq_a c\}.$$

The relevance of $A(\Phi_+)$ in representation theory. Let Λ be a Dynkin algebra of type Δ. There are bijections between:

- The set $NC(\Delta)$ of **noncrossing partitions** of type Δ
- The multiplicity-free **support tilting** modules T in mod Λ
  (a Λ-module T is support-tilting iff it is a tilting $\Lambda/(\mathrm{Ann}(T))$-module)
- The **torsion classes** in mod Λ
  (the full subcategories closed under extensions and factor modules).
- The **thick** (or wide) subcategories in mod Λ
  (The full exact abelian subcategories closed under extensions).
- The **antichains** in mod Λ
  (An antichain is a set of pairwise orthogonal indecomposable modules.)

These bijections are due to various authors; the relationship between non-crossing partitions and representations of quivers was first noticed by Fomin-Zelevinksy. The decisive reference to thick subcategories is due to Ingalls-Thomas (2009).

Recall: For a Dynkin algebra Λ of type Δ, there are bijections between:

- The set $NC(\Delta)$ of **noncrossing partitions** of type Δ
- The multiplicity-free **support tilting** modules T in mod Λ
- The **torsion classes** in mod Λ
- The **thick** subcategories $\mathcal{C}$ in mod Λ
- The **antichains** in mod Λ

Here are some of these correspondences: The modules generated by a support tilting module T or by a thick subcategory $\mathcal{C}$ yield the corresponding torsion class. The set of simple objects of a thick subcategory is an antichain. The modules filtered by modules in an antichain $\mathcal{A}$ form a thick subcategory.

Note that Λ determines an orientation of Δ, thus a Coxeter element c in the Weyl group W. If X is an indecomposable Λ-module with dimension vector x, then the reflection $\rho_X = \rho_x$ belongs to W. If $T = T_1 \oplus \cdots \oplus T_t$ is multiplicity-free and support-tilting, we may assume that $\mathrm{Hom}(T_i, T_j) = 0$ for $i > j$. Then $\rho_{T_1} \cdots \rho_{T_t} \leq_a c$, thus $\rho_{T_1} \cdots \rho_{T_t}$ belongs to $NC(W, c)$.

Recall again: For a Dynkin algebra Λ of type Δ, there are bijections between:

- The set $NC(\Delta)$ of **noncrossing partitions** of type Δ
- The **torsion classes** in mod Λ
- The **thick** subcategories in mod Λ
- The **antichains** in mod Λ

The sets:

$\{$torsion classes$\}$, $\{$thick subcategories$\}$, $\{$antichains$\}$

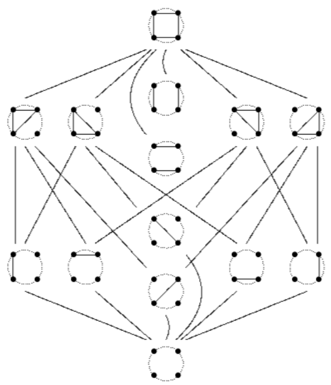
are in a natural way posets (using the inclusion orders).

Thus we obtain different natural poset structures for $NC(\Delta)$.

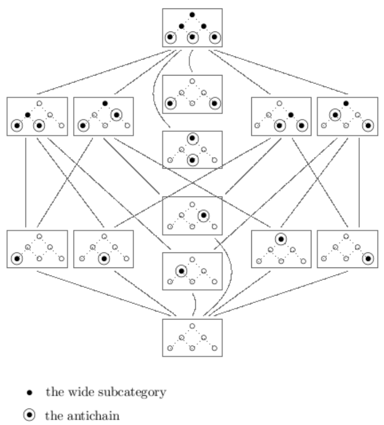
The absolute order on $NC(\Delta)$ corresponds to the thick-subcategory-order. Since the thick subcategories form a lattice, this provides a unified proof that $NC(\Delta)$ is a lattice.

The case $\mathbb{A}_3$

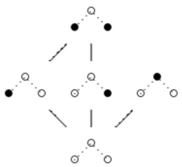
The absolute order on $NC(\Delta)$



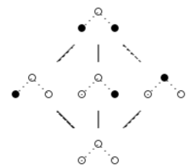
The thick subcategories ordered:



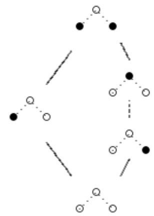
Different orders on the set of antichains in $\operatorname{mod}\Lambda$ where Λ is a Dynkin algebra of type $\mathbb{A}_2$



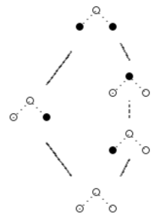
inclusion order



wide subcategory order



torsion class order



torsionfree class order

The relevance of $A(\Phi_+)$ in representation theory. II.

Recall: For a Dynkin algebra Λ of type Δ, there is a bijection between:

- The set $NC(\Delta)$ of **noncrossing partitions** of type Δ.
- The **support tilting modules** T in mod Λ.

Thus, according to Buan-Marsh-Reineke-Reiten-Todorov, also to

- The **cluster tilting objects** in the cluster category of type Δ. (thus to the clusters in the Fomin-Zelevinsky cluster complex)

The cluster category of type Δ is the orbit category $D^b(\text{mod}\,\Lambda)/F$ with $F = \tau^-[1]$; its indecomposable objects correspond bijectively to the "almost positive" roots, this is the set $\Phi_{\geq -1}$ consisting of the positive and the negative simple roots.

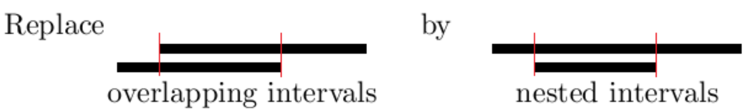
Non-crossing versus non-nesting:

Aim: to find a bijection

$$\begin{array}{c}\{\text{non-nesting}\} \leftrightarrow \{\text{non-crossing}\} \\ \{\text{antichains in } \Phi_+\} \leftrightarrow \{\text{antichains in mod } \Lambda\}\end{array}$$

Observe: the left hand side does not depend on the choice of an orientation of Δ, whereas the right hand side does.

In type $\mathbb{A}_n$ with linear orientation, there is a natural bijection:



The first uniform (but still complicated) bijection for general Δ was constructed by Armstrong-Stump-Thomas (2011).

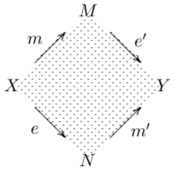
Non-crossing versus non-nesting:

Aim: to find a natural bijection using representation theory.

{non-nesting} ↔ {non-crossing}
{antichains in Φ_+} ↔ {antichains in mod Λ}

It seems that one can define in general a natural bijection between anti-pairs in mod Λ and anti-pairs in the root poset using "crossovers" (this is proven for all types but $\mathbb{E}_7$ and $\mathbb{E}_8$, but not yet published).

A **simple crossover** is a pullback-pushout diagram with m, m' mono, and e, e' epi,
X, Y an antipair in Φ_+ (non-nesting)
M, N antipair in mod Λ (non-crossing)



Hidden symmetries III:

The maximal chains in the antichain lattice of Φ_+.

The set $C(A(\Phi_+))$ of maximal chains in $A(\Phi_+)$

It has the following cardinality:

A_n	D_n	B_n, C_n	E_6	E_7	E_8	F_4	G_2
$(n+1)^{n-1}$	$2(n-1)^n$	n^n	41 472 $= 2^9 3^4$	1 062 882 $= 2.3^{12}$	37 968 750 $= 2.3^5 5^7$	432 $=2^4 3^3$	6 $=2.3$

Here is a formula as exhibited by Chapoton in 2006

$$|C(A(\Phi_+))| = n!h^n/|W| = h^n/(x_1 \ldots x_n x_0).$$

$(x_1, \ldots, x_n)$ is the maximal root and $x_0 = 1 + |\{1 \leq i \leq n \mid x_i = 1\}|$ (and this is the index of the root lattice in the weight lattice).

The maximal chains correspond to

factorization of a Coxeter element using reflections.

For these factorizations, the numbers were determined by Deligne-Tits-Zagier (letter of Deligne to Looijenga, 1974)

The set $C(A(\Phi_+))$ of maximal chains in $A(\Phi_+)$

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Λ a Dynkin algebra of type Δ. Krause-Hubery have pointed out: Under the Ingalls-Thomas bijection, the maximal chains in $A(\Phi_+(\Delta))$ correspond bijectively to the **complete exceptional sequences** in mod Λ.

An exceptional sequence in mod Λ is a sequence $(M_1, \dots, M_t)$ of indecomposable modules M_i such that $\mathrm{Hom}(M_i, M_j) = 0 = \mathrm{Ext}^1(M_i, M_j)$ for $i > j$. An exceptional sequence is complete provided $t = n$.

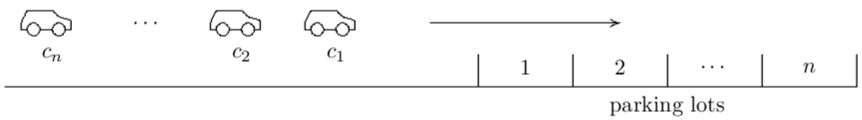
The number of complete exceptional sequences in the quiver case was directly calculated by Seidel (2002), in general by Obaid-Nauman-Shammakh-Fakieh-Ringel (2013).

The case $\mathbb{A}_n$: $|C(A(\Phi_+(\mathbb{A}_n)))| = (n+1)^{n-1}$.

Here are some related counting problems:

- The number of **labeled trees** with n+1 vertices is $(n+1)^{n-1}$ (Sylvester 1857, Borchardt 1860, Cayley 1889).
- **Shi arrangements:** The hyperplanes $x_i - x_j = 0$ or 1 for all $i < j$ divide the n-space in $(n+1)^{n-1}$ connected components (Shi 1989).
- The number of **parking functions** for n cars is $(n+1)^{n-1}$ (Pyke 1959).

Parking functions



An endofunction f of $\{1, 2, \dots, n\}$ is called a **parking function** for n cars if $|\{x \mid f(x) \le i\}| \ge i$ for all i (if one uses the rule that the car c_i takes the parking lot $f(i)$ in case it is free, otherwise the next free parking lot, then any car finds a parking lot).

Theorem (Pyke 1959). *There are* $(n+1)^{n-1}$ *parking functions for* n *cars.*

Example: the parking functions for 2 cars are $(f(1), f(2)) = (1,1), (1,2), (2,1)$.

Stanley (1997) exhibited a bijection between the parking functions for n cars and the maximal chains of non-crossing partitions of type $\mathbb{A}_n$.

Theorem. *Let Δ be a Dynkin diagram with n vertices. The braid group*

$$B_n = \langle\, s_i \mid 1 \le i < n, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, s_i s_j = s_j s_i \ \text{for}\ |i-j| = 2 \rangle$$

operates transitively on $C(A(\Phi_+(\Delta)))$.

Deligne-Tits-Zagier (for factorizations of a Coxeter element),
Crawley-Boevey, Ringel (for complete exceptional sequences).

Summary.

By looking at the subset Φ_+ of the root system Φ (or also at $\Phi_{\geq -1}$) one **destroys** all the natural symmetries of Φ.

However one obtains many **new symmetries** by looking

- at Φ_+ itself,
- at the set $A(\Phi_+)$ of antichains in Φ_+, or
- the set $C(A(\Phi_+))$ of maximal chains in $A(\Phi_+)$.

Many symmetries of the positive roots of a root system of type Δ are encoded in the corresponding lattice of non-crossing partitions.

$*\ *\ *$

But be aware: there are addional symmetries of Φ_+ and $\Phi_{\geq -1}$ not covered in the lecture, for example several given by the cluster complex ...

Also, the (finite) root systems are special cases of the general Kac-Moody root systems, and many of the presented results can be generalized to this setting.

Symmetry

is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection.

H.Weyl (Symmetry)

Symmetry

is ennui,
and ennui is the very essence of grief and melancholy.

Victor Hugo (Les Miserables)