A partial history of the Schur functor

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Outline

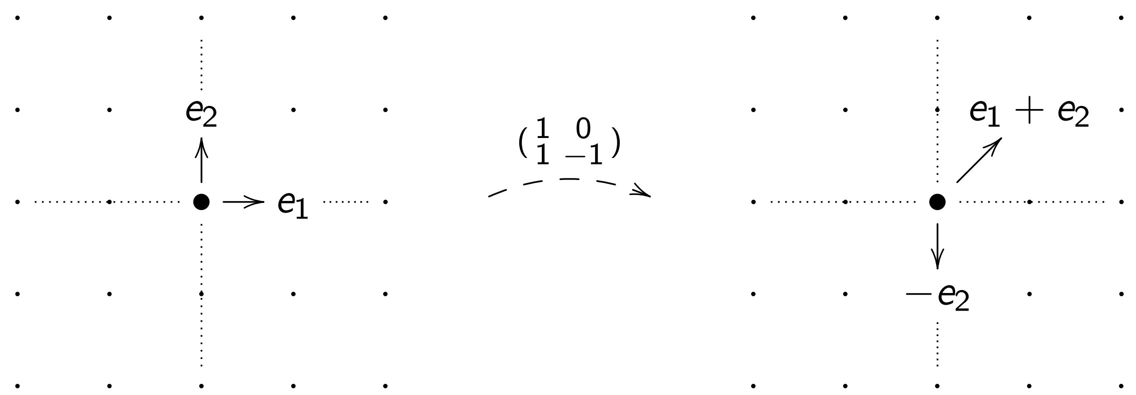
- Issai Schur (1875–1941) obtained his doctorate from the University of Berlin in 1901. His thesis helped to found the field of representation theory. In it, he described the polynomial representations of the general linear group $GL_n(\mathbb{C})$, in particular the invariant subspaces of tensor space.
- One of Schur's main ideas, now called the Schur functor, relates representations of $GL_n(\mathbb{C})$ to representations of the symmetric group S_d.
- This idea has had a large influence on the development of representation theory up to the present day, and regularly reappears in different guises, most recently in geometric modular representation theory, a new approach with the potential to solve long-standing problems.

Invariant subspaces of tensor space

Let F be an infinite field, e.g. $\mathbb{R}$ or $\mathbb{C}$, and let $n \in \mathbb{Z}^+$.
The vector space $F^n = \{(a_1, \cdots, a_n) \mid a_i \in F\}$ has a standard basis

$$\{e_1 = (1, 0, \cdots, 0), e_2 = (0, 1, 0, \cdots, 0), \cdots, e_n = (0, \cdots, 0, 1)\}.$$

An arbitrary basis is obtained from this one by applying a linear transformation $e_j \mapsto g.e_j = \sum_i a_{ij} e_i$, where $g = (a_{ij}) \in GL_n(F)$, the general linear group of invertible $n \times n$ matrices over F.



For any $d \in \mathbb{Z}^+$ we can form the tensor space $T^d(F^n)$ with basis

$$\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} \mid i_1, i_2, \cdots, i_d \in \{1, \cdots, n\}\}.$$

For $g \in GL_n(F)$, we define a linear transformation L_g of $T^d(F^n)$ by

$$L_g(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}) = g.e_{i_1} \otimes g.e_{i_2} \otimes \cdots \otimes g.e_{i_d},$$

where the right-hand side is expanded using the distributive law.
This makes $T^d(F^n)$ a representation of the group $GL_n(F)$.

Question

Which subspaces V of $T^d(F^n)$ are invariant in the sense that $L_g(V) = V$ for all $g \in GL_n(F)$?

The point of the question is that only the invariant subspaces of $T^d(F^n)$ are ‘intrinsic’, i.e. independent of the basis of F^n.

Example ($d = 2$)

$T^2(F^n)$ has basis $\{e_i \otimes e_j\}$. If $n \geq 2$, the 1-dimensional subspace $\text{span}(e_1 \otimes e_1)$ is not invariant. If an invariant subspace contains $e_1 \otimes e_1$, it has to contain $e_i \otimes e_i$ for any i, and also

$$(e_i + e_j) \otimes (e_i + e_j) - e_i \otimes e_i - e_j \otimes e_j = e_i \otimes e_j + e_j \otimes e_i.$$

This leads to one example of an invariant subspace, the space of symmetric tensors:

$$\mathrm{Sym}^2(F^n) = \text{span}\{e_i \otimes e_i,\ e_i \otimes e_j + e_j \otimes e_i\}.$$

It is fairly easy to show that the only other nontrivial invariant subspace of $T^2(F^n)$ is the space of alternating tensors:

$$\mathrm{Alt}^2(F^n) = \text{span}\{e_i \otimes e_j - e_j \otimes e_i\}.$$

For $\sigma \in S_d$, the symmetric group of permutations of $\{1, 2, \cdots, d\}$, define the linear transformation P_σ of $T^d(F^n)$ by

$$P_\sigma(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d}) = e_{i_{\sigma^{-1}(1)}} \otimes e_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(d)}}.$$

Thus $T^d(F^n)$ is also a representation of the group S_d.
Schur's crucial observation is that any P_σ commutes with any L_g:

$$\begin{aligned} L_g(P_\sigma(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d})) &= g.e_{i_{\sigma^{-1}(1)}} \otimes g.e_{i_{\sigma^{-1}(2)}} \otimes \cdots \otimes g.e_{i_{\sigma^{-1}(d)}} \\ &= P_\sigma(L_g(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d})). \end{aligned}$$

One consequence is that any subspace of $T^d(F^n)$ defined in terms of the P_σ's is automatically invariant for the L_g's.

Example ($d = 2$)

$\mathrm{Sym}^2(F^n) = \ker(\mathrm{id} - P_{(1\,2)})$, $\mathrm{Alt}^2(F^n) = \mathrm{im}(\mathrm{id} - P_{(1\,2)})$.

The Schur functor

This is a functor $\Phi : \mathrm{Pol}_d(GL_n(F)) \to \mathrm{Rep}(S_d, F)$ where

- $\mathrm{Pol}_d(GL_n(F))$ is the category of polynomial representations of $GL_n(F)$ of degree d, e.g. invariant subspaces of $T^d(F^n)$;
- $\mathrm{Rep}(S_d, F)$ is the category of representations of S_d over F.

For $V \in \mathrm{Pol}_d(GL_n(F))$, we define

$$\begin{aligned}\Phi(V) &= \mathrm{Hom}_{GL_n(F)}(T^d(F^n), V)\\ &= \{\text{linear maps } f : T^d(F^n) \to V \text{ commuting with each } L_g\}\end{aligned}$$

with the action of $\sigma \in S_d$ given by $f \mapsto f \circ P_{\sigma^{-1}}$.

Example ($d = 2$)

$\Phi(\mathrm{Sym}^2(F^n)) = \mathrm{span}(\mathrm{id} + P_{(1\,2)}), \Phi(\mathrm{Alt}^2(F^n)) = \mathrm{span}(\mathrm{id} - P_{(1\,2)})$.

Schur's advisor Frobenius had already described $\mathrm{Rep}(S_d, \mathbb{C})$, and the same description holds whenever $d! = |S_d|$ is invertible in F, i.e. $\mathrm{char}(F) \notin \{2, \cdots, d\}$.

- The irreducible representations (i.e. those without S_d-invariant subspaces) are parametrized by partitions of the number d, i.e. expressions $d = d_1 + d_2 + \cdots$ with $d_i \in \mathbb{N}$, $d_1 \geq d_2 \geq \cdots$.
- Every representation is uniquely a direct sum of irreducibles.

Theorem (Schur 1901)

Assume $\mathrm{char}(F) \notin \{2, \cdots, d\}$. (Actually Schur assumed $F = \mathbb{C}$.)

1. *The functor Φ identifies $\mathrm{Pol}_d(GL_n(F))$ with the subcategory of $\mathrm{Rep}(S_d, F)$ given by partitions of d of length $\leq n$.*
   In particular, if $d \leq n$, then Φ is an equivalence.
2. *There is an irreducible invariant subspace $V^{d_1, \cdots, d_n} \subset T^d(F^n)$ for any partition of d of length $\leq n$. Every invariant subspace is a direct sum of isomorphic copies of $V^{d_1, \cdots, d_n}$'s.*

These Weyl modules $V^{d_1,\cdots,d_n}$ are well understood, with explicit defining equations and bases. At the two extremes we have:

$$V^{d,0,\cdots,0} = \mathrm{Sym}^d(F^n) = \{x \in T^d(F^n) \,|\, P_\sigma(x) = x \text{ for all } \sigma \in S_d\},$$
$$V^{1,1,\cdots,1} = \mathrm{Alt}^d(F^n) = \mathrm{im}\{\sum_{\sigma \in S_d} \mathrm{sign}(\sigma) P_\sigma\} \quad (\text{if } d \leq n).$$

Example ($d = 3$, $n \geq 2$)

$$V^{2,1,0,\cdots,0} = \mathrm{im}(\mathrm{id} - P_{(1\,2)}) \cap \ker(\mathrm{id} + P_{(1\,2\,3)} + P_{(1\,3\,2)})$$
$$= \mathrm{span}\{e_i \otimes e_j \otimes e_i - e_j \otimes e_i \otimes e_i,$$
$$e_i \otimes e_j \otimes e_k - e_j \otimes e_i \otimes e_k + e_k \otimes e_j \otimes e_i - e_j \otimes e_k \otimes e_i\}$$

Theorem (Weyl 1925)

$$\dim V^{d_1,\cdots,d_n} = \prod_{1 \leq i < j \leq n} \frac{d_i - d_j + j - i}{j - i}.$$

Some subsequent developments

Schur's classification of irreducible polynomial representations of $GL_n(\mathbb{C})$ inspired analogous classifications of representations of:

- semisimple Lie groups and Lie algebras (Weyl 1920s/30s);
- algebraic groups (Chevalley, Borel 1950s/60s);
- certain infinite-dimensional Lie algebras (Kac, Moody 1970s);
- quantum groups (Drinfel'd, Lusztig et al. 1980s/90s).

The relationship between the commuting actions of $GL_n(\mathbb{C})$ and S_d on $T^d(\mathbb{C}^n)$, now known as Schur–Weyl duality, has been extended to all of these contexts, leading to analogues such as:

$GL_n(\mathbb{C})$	S_d
$O_n(\mathbb{C})$ (orthogonal group)	Br_d (Brauer algebra)
quantum $GL_n(\mathbb{C})$	$\mathcal{H}_d$ (Hecke algebra)
$\vdots$	$\vdots$

The modular case

If $\mathrm{char}(F)$ is a prime $p \leq d$, $\mathrm{Rep}(S_d, F)$ is not so well understood. The Schur functor

$$\Phi : \mathrm{Pol}_d(GL_n(F)) \to \mathrm{Rep}(S_d, F)$$

is still crucial, but information now also flows from left to right.

- We still have $V^{d_1,\cdots,d_n}$ and Weyl's formula for dim $V^{d_1,\cdots,d_n}$.
- However, $V^{d_1,\cdots,d_n}$ is usually reducible.
- $V^{d_1,\cdots,d_n}$ always has a unique irreducible quotient $L^{d_1,\cdots,d_n}$.
- However, we do not have a general formula for dim $L^{d_1,\cdots,d_n}$.
- Φ is still an exact functor of abelian categories.
- However, it is no longer faithful: we can have $\Phi(L^{d_1,\cdots,d_n}) = 0$.

Example ($p = d = 2 \leq n$)

Since $-1 = 1$ in F,

$$\begin{aligned} \mathrm{Alt}^2(F^n) &= \mathrm{span}\{e_i \otimes e_j + e_j \otimes e_i\} \\ \subset \mathrm{Sym}^2(F^n) &= \mathrm{span}\{e_i \otimes e_i,\ e_i \otimes e_j + e_j \otimes e_i\}, \end{aligned}$$

so $V^{2,0,\cdots,0} = \mathrm{Sym}^2(F^n)$ is reducible. We have

$$L^{2,0,\cdots,0} = \mathrm{Sym}^2(F^n)/\mathrm{Alt}^2(F^n).$$

This irreducible representation is 'not seen' by the Schur functor:

$$\Phi(\mathrm{Sym}^2(F^n)) = \Phi(\mathrm{Alt}^2(F^n)) = \mathrm{span}(\mathrm{id} + P_{(1\,2)}),$$
$$\text{so } \Phi(L^{2,0,\cdots,0}) = 0.$$

The Lusztig conjecture

Fix n and $p = \mathrm{char}(F)$ and let $d_1, \cdots, d_n$ (and hence d) vary.

1979: Lusztig defined a function $f(d_1, \cdots, d_n)$ combinatorially and conjectured, on the basis of computations and analogies, that $\dim L^{d_1,\cdots,d_n} = f(d_1, \cdots, d_n)$ as long as $p \geq 2n - 3$.

1994: Andersen–Jantzen–Soergel, completing a program begun by Kazhdan–Lusztig, proved that Lusztig's conjecture is true as long as $p \gg n$.

2013: Williamson, building on work of Soergel, has found a family of counterexamples to Lusztig's conjecture showing that no polynomial lower bound $p \geq P(n)$ is sufficient. His calculations use geometric modular representation theory.

Geometric interpretation of the Schur functor

Let $\mathcal{N}_d$ be the set of all nilpotent $d \times d$ complex matrices X. The group $GL_d(\mathbb{C})$ acts on $\mathcal{N}_d$ by conjugation. By the Jordan form theorem, the orbits are in bijection with partitions of d: the orbit $\mathcal{O}_{d_1,d_2,\cdots}$ consists of matrices with Jordan blocks of sizes $d_1, d_2, \cdots$.

Example ($d = 2$)

$\mathcal{N}_2 = \{\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \mid a, b, c \in \mathbb{C}, a^2 + bc = 0\}$ is the union of two orbits, $\mathcal{O}_{1,1,0,\cdots} = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$ and $\mathcal{O}_{2,0,\cdots} = \mathcal{N}_2 \setminus \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.

Theorem (Mautner 2010, using Mirković–Vilonen 2007)

When $d \leq n$, $\mathrm{Pol}_d(GL_n(F))$ is equivalent to $\mathrm{Perv}(\mathcal{N}_d, F)$, the category of $GL_d(\mathbb{C})$-equivariant perverse sheaves on $\mathcal{N}_d$ with coefficients in F, in such a way that $L^{d_1,\cdots,d_n}$ corresponds to the intersection cohomology complex $IC(\overline{\mathcal{O}_{d_1,\cdots,d_n}}, F)$.

There was already a well-known definition of a functor $\mathbb{S} : \mathrm{Perv}(\mathcal{N}_d, F) \to \mathrm{Rep}(S_d, F)$, the Springer correspondence:

- take Fourier transform to produce a sheaf on $\mathrm{Mat}_d(\mathbb{C})$;
- restrict to the set $\mathrm{Mat}_d(\mathbb{C})_{\mathrm{rs}}$ of matrices with distinct eigenvalues, producing a locally constant sheaf;
- thus obtain a monodromy representation of $\pi_1(\mathrm{Mat}_d(\mathbb{C})_{\mathrm{rs}})$;
- this factors through the quotient map $\pi_1(\mathrm{Mat}_d(\mathbb{C})_{\mathrm{rs}}) \twoheadrightarrow S_d$.

That this works was proved by Springer (1976) when $F = \mathbb{C}$ and Juteau (2007) and Mautner (2010) in general.

Theorem (Mautner, Achar–H.–Juteau–Riche)

When $d \leq n$, the Schur functor is the composition of:

- *Mautner's equivalence $\mathrm{Pol}_d(GL_n(F)) \xrightarrow{\sim} \mathrm{Perv}(\mathcal{N}_d, F)$;*
- *the Springer correspondence $\mathbb{S} : \mathrm{Perv}(\mathcal{N}_d, F) \to \mathrm{Rep}(S_d, F)$;*
- *tensoring with the sign character.*

Let G be any split connected reductive algebraic group over F. Let T be a maximal torus of G (e.g. diagonal matrices in $GL_n(F)$), and let $W = N_G(T)/T$ be the Weyl group (e.g. S_n for $GL_n(F)$). Define a functor

$$\Phi' : \mathrm{Rep}(G, F) \to \mathrm{Rep}(W, F) : V \mapsto V^T \quad (T\text{-fixed vectors}).$$

This generalizes the $d = n$ case of the Schur functor:

$$\Phi(V) \cong \Phi'(V \otimes \det{}^{-1}) \otimes \mathrm{sign} \quad \text{for } V \in \mathrm{Pol}_n(GL_n(F)).$$

Theorem (Achar–H.–Riche)

Let $\mathcal{N}$ be the nilpotent cone of the Langlands dual group $G^\vee(\mathbb{C})$. Restricted to a suitable subcategory $\mathrm{Rep}(G, F)_{\mathrm{sm}}$, Φ' is the composition of:

- *a certain functor $\mathrm{Rep}(G, F)_{\mathrm{sm}} \to \mathrm{Perv}(\mathcal{N}, F)$;*
- *the Springer correspondence $\mathbb{S} : \mathrm{Perv}(\mathcal{N}, F) \to \mathrm{Rep}(W, F)$.*