THE UNIVERSITY OF SYDNEY

Sydney University Mathematical Society Problem Competition 2014

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Those who enter do so as individuals, and must **not** receive help with the problems, e.g. from fellow students, lecturers or online groups.

Separate prizes ($75 book vouchers from the Co-op) will be awarded for the best solution to each of the eight problems, and entrants may submit solutions to whichever problems they wish. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry; there is one Norbert Quirk Prize for each of 1st, 2nd, 3rd and Honours years (the prize for Honours entrants is new in 2014).

The problems are listed in roughly increasing order of difficulty. For the easier problems, solutions which include an interesting extension or generalization will be rated more highly. For the harder problems, partial solutions may be accepted. If two or more solutions to a problem are essentially equal, the prize will be given to the student(s) in the earlier year of university.

Entries must be received by **Thursday, August 14, 2014**. They may be posted to Associate Professor Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or handed in to Carslaw room 805. Please mark your entry SUMS Problem Competition 2014, and include your name, university, student number, year of study, and postal address (or email address for University of Sydney students) for the awarding of prizes. Please note that entries will not be returned.

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1. Let $A_1, A_2, \cdots, A_{2n}$ be the vertices of a convex $(2n)$-gon in the plane, listed in clockwise order (here $n \geq 3$). Suppose that opposite edges of the $(2n)$-gon are parallel (that is, A_1A_2 is parallel to $A_{n+2}A_{n+1}$, A_2A_3 is parallel to $A_{n+3}A_{n+2}$, and so on until A_nA_{n+1} is parallel to A_1A_{2n}). Prove that the main diagonals $(A_1A_{n+1}, A_2A_{n+2}, \cdots, A_nA_{2n})$ intersect in a single point if and only if opposite edges of the $(2n)$-gon have equal length.

2. Anna is playing a mathematical computer game. The computer is hiding a polynomial P; the degree and coefficients of P are unknown to Anna, but she does know that the coefficients are strictly positive real numbers. In each move, Anna inputs a real number a and the computer outputs $P(a)$. This is repeated until Anna can determine what P must be.

For any strategy S used by Anna, denote by $S(P)$ the number of moves needed to determine P. Call a strategy S *optimal* if $S(P) \leq S'(P)$ for all possible strategies S' and all polynomials P with strictly positive real coefficients. Does there exist an optimal strategy?

3. For any positive integer n, let $P_n(x)$ be the polynomial defined by

$$P_n(x) = x^n + 2x^{n-1} + 3x^{n-2} + \cdots + nx + (n+1).$$

Show that, if either $n+1$ or $n+2$ is prime, then $P_n(x)$ is irreducible (that is, it cannot be written as a product of two non-constant polynomials with integer coefficients).

4. a) Show that the zero function is the only continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_a^{a^2} f(x)\,dx = 0, \quad \text{for all } a \in \mathbb{R}.$$

b) Show that there exists a nonzero continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_a^{a^2+1} f(x)\,dx = 0, \quad \text{for all } a \in \mathbb{R}.$$

5. Find all pairs (a, b) of positive integers such that $2a$ divides $b^2 + 1$ and b divides $2a^2 + 1$.

6. For which connected finite simple graphs can one label each vertex v with a positive integer $f(v)$ in such a way that, for every v, the sum of the labels of the vertices adjacent to v is $2f(v) - 1$?

7. Say that a function $f : \mathbb{Z} \to \{1, 0, -1\}$ is a *perturbed sign function* if it satisfies the following properties for $a \gg 0$ (i.e., there is some $N \geq 0$ such that these equations hold for all $a \geq N$):

$$f(a) = 1, \quad f(-a) = -1, \quad \sum_{i=-a}^{a} f(i) = 0.$$

Given such a perturbed sign function f, let its *degeneracy* $d(f)$ be the number of $a \in \mathbb{Z}$ such that $f(a) = 0$, and let its *weight* $w(f)$ be the number of pairs $(a, b) \in \mathbb{Z}^2$ such that $a < b$ and $f(a) > f(b)$. Find a formula for the power series $\sum x^{w(f)}$ where the sum is over all perturbed sign functions f of fixed degeneracy d.

8. Let n denote a positive integer. The symmetric group S_n is the group of permutations of the set $\{1, 2, \cdots, n\}$. This group acts naturally on the set $\mathcal{P}_n$ of all subsets of $\{1, 2, \cdots, n\}$: if $\sigma \in S_n$ and $I \in \mathcal{P}_n$, then $\sigma(I)$ has the usual meaning of $\{\sigma(i) \mid i \in I\}$. We regard S_n as a subgroup of S_{n+1} or of S_{n+2} in the obvious way, i.e. S_n is identified with the subgroup of S_{n+1} consisting of permutations that fix $n + 1$, and with the subgroup of S_{n+2} consisting of permutations that fix $n + 1$ and $n + 2$.

a) Show that the action of S_n on $\mathcal{P}_n$ extends to S_{n+1} (that is, there is an action of S_{n+1} on $\mathcal{P}_n$ which, when restricted to S_n, coincides with the aforementioned action of S_n on $\mathcal{P}_n$).

b) For which n does the action of S_n on $\mathcal{P}_n$ extend to S_{n+2}?