Sydney University Mathematical Society Problem Competition 2009

This competition is open to undergraduates (including Honours students) at any Australian university or tertiary institution. Entrants may use any source of information except other people. The problems will also be posted on the web page **http://www.maths.usyd.edu.au/u/SUMS/**.

Entrants may submit solutions to as many problems as they wish. Prizes ($60 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Students from the University of Sydney are also eligible for the Norbert Quirk Prizes, based on the overall quality of their entry (one for each of 1st, 2nd and 3rd years). Extensions and generalizations of any problem are invited and are taken into account when assessing solutions. If two or more solutions to a problem are essentially equal, preference may be given to students in the earlier year of university; otherwise, prizes may be shared. If a problem receives no correct solutions, its prize-money will be redistributed among the other problems.

Entries must be received by **Friday, August 14, 2009**. They may be posted to Dr Anthony Henderson, School of Mathematics and Statistics, The University of Sydney, NSW 2006, or delivered in person to Room 805, Carslaw Building. Please mark your entry SUMS Problem Competition 2009, and include your name, university, student number, year of study, and postal address (or email address in the case of University of Sydney students) for the return of your entry and prizes.

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1. The sisters Alice, Bess, and Cath have become proficient at factorizing numbers, so their father David invents a puzzle for them. He chooses three secret integers a, b, c, all greater than 1, and then puts a sticker on Alice's forehead showing the number bc (the product of b and c), one on Bess' forehead showing ac, and one on Cath's forehead showing ab. Each of the girls can see her sisters' stickers but not her own, and must try to work out the number on her own sticker, knowing how the numbers were derived. After a few seconds' thought, Alice says smugly "I know my number". Bess then says "I wasn't sure about my number at first, but knowing that Alice knows hers, I know mine". Even after hearing her sisters' comments, Cath can't work out the number on her sticker; but when David gives her the hint that it is even, she can. What is a?

2. In this problem, a *word* is a finite string of capital letters (not necessarily meaningful in English) in which no letter occurs in two consecutive positions. Thus AFARSFA and BEGEB are words, but ABBA is not. A word is *palindromic* if, like BEGEB, it reads the same backwards as forwards (a single letter counts as a palindromic word, but the empty word does not). We say that a word W is *contained* in another word W' if the letters of W occur in the right order among the letters of W', not necessarily consecutively. For instance, AFRFA, S and FASF are all contained in AFARSFA (as is AFARSFA itself), but SRA is not. Prove that if a word W is n letters long, there are at least n palindromic words which are contained in W.

3. A *repeating number* is a positive integer whose decimal expression consists of two or more occurrences of the same block of digits: examples are 44, 575757, and 616616. Show that there is no repeating number whose square is also a repeating number.

4. Let $a_1, a_2, a_3, \cdots$ be positive real numbers such that $\sum_{n=1}^{\infty} a_n = 1$. Show that $\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n}$ converges to a value strictly less than e.

5. Let A_n be the $n \times n$ matrix whose (i, j)-entry is 1 if $n \le i + j \le n + 1$ and zero otherwise. Find the eigenvalues of A_n.

6. Peg solitaire is sometimes played with an array of pegs which form an equilateral triangle, except that initially there is one position left empty. A move consists of jumping a peg over an adjacent peg into an empty position on the other side, where the line of motion is parallel to one of the sides of the triangle; the peg which was jumped over is then removed. The aim is to have only one peg remaining at the end. (Videos of such a 'Peg Puzzle' can be found online.)

By contrast, the four-dimensional beings in the neighbouring universe play solitaire with an array of pegs forming a regular tetrahedron, initially with one peg missing. A move now affects four consecutive positions rather than three: it consists of jumping a peg over an adjacent peg *and* over a third peg on the other side of that, into an empty position on the other side of the third peg, where the line of motion is parallel to one of the edges of the tetrahedron; *both* pegs which were jumped over are then removed. The aim is to have only one peg or two adjacent pegs remaining at the end. Show that if the initial empty position is in the exact centre of the tetrahedron, this aim cannot be achieved.

7. Define a sequence of integers $a_0, a_1, a_2, \cdots$ by the initial condition $a_0 = 1$ and the recurrence relation $a_n = \sum_{k=1}^{n} \binom{n-1}{k-1} k!\, a_{n-k}$ for $n \ge 1$. Prove that $a_n - 1$ is always a multiple of n.

8. A regular polygon may be defined as a convex polygon whose vertices all lie on a circle and whose edges all have the same length. A *semi-regular* polygon is a convex polygon which has an even number of vertices all lying on a circle, such that the lengths of its edges, in clockwise order, are $a, b, a, b, \cdots, a, b$ for some $a \ne b$. (For instance, a non-square rectangle is semi-regular.) Prove that, given any regular polygon P, it is possible to construct with straightedge and compass a semi-regular polygon Q which has the same perimeter-length as P and encloses the same area as P.

9. Let $F = \{0, 1, \cdots, p-1\}$ be the field of integers modulo a prime $p \ne 2$. Let X be a nonempty subset of $F^d = \{(x_1, x_2, \cdots, x_d) \mid x_i \in F\}$ for some positive integer d. Prove that there exist $a_1, a_2, \cdots, a_d, b \in F$ such that the equation $a_1 x_1 + a_2 x_2 + \cdots + a_d x_d = b$ has an odd number of solutions $(x_1, x_2, \cdots, x_d)$ in X.

10. For any positive integer n, prove that

$$\sum_{a=0}^{n} \binom{n}{a} a^{n-a} (n-a)^a \le \frac{1}{2}\, n^n.$$