Sydney University Mathematical Society
Problems Competition 2001

This competition is open to all undergraduates at any Australian university or tertiary institution. Prizes ($50 book vouchers from the Co-op Bookshop) will be awarded for the best correct solution to each of the 10 problems. Entrants from the University of Sydney will also be eligible for the Norbert Quirk Prizes (one for each of 1st, 2nd and 3rd years). Entries from fourth year students will be considered. When prizewinners are being selected, if two or more entries to a problem are essentially equal, then preference may be given to the students in the earlier year of university.

Contestants may use any source of information except other people. Solutions are to be received by 4.00 pm on Friday, September 28, 2001. They may be given to Dr. Donald Cartwright, Room 620, Carslaw Building, or posted to him at the School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006. Entries must state name, university, student number, course and year, term address and telephone number, and be marked **2001 SUMS Competition**. The prizes will be awarded towards the end of the academic year.

The SUMS committee is grateful to all those who have provided problems. We are always keen to get more. Send any, with solutions, to Dr. Cartwright, at the above address.

Problems

(Extensions and generalizations of any problem are invited and are taken into account when assessing solutions.)

1. Linda is staying with her very rich Uncle Bruce and even richer Aunt Joylene. Uncle Bruce loves dollar coins and 10 cent pieces but hates 20 cent and 50 cent coins, of which he has a vast supply. He tells Linda he will give her two 20 cent coins and one 50 cent coin for every dollar coin she gives him, and for every 10 cent coin he will give her a 50 cent coin. Aunt Joylene prefers 20 and 50 cent coins to 10 cent coins and dollars. She says she will give Linda a dollar coin and two 10 cents for every 50 cent coin, and a dollar coin for every 20 cent coin. Linda amuses herself by first taking a single dollar coin and swapping it with Bruce, she then takes the proceeds to Joylene and swaps them for dollars and ten cent coins. Then she goes back to Bruce, and so on. After n visits to Joylene, how much money will she have?

2. Twelve delegates attend a conference at which they are seated around a circular table. Each delegate brings with him to the conference three copies of a document which he has prepared. He gives one copy to each of his two immediate neighbours at the table, and keeps the third copy himself. While the delegates are away at coffee, having left the papers in question on their chairs, a thief rushes into the room and snatches one document at random from each chair. What is the probability that the thief obtains a complete set of the twelve documents?

3. Suppose that two rectangular boxes B and B' have length, breadth and height ℓ, b and h, and ℓ', b' and h', respectively. Suppose that $B \subset B'$. Show that $\ell+b+h \leq \ell'+b'+h'$.

4. Suppose that $a > 0$ is an integer, and that $4n(n+1)a^2+1$ is a perfect square for all integers $n \geq 0$. Show that $a = 1$. Now let $\alpha(n) = an + b$, where $a, b \geq 0$ are integers. Suppose that $4n(n+1)\alpha(n)^2 + 1$ is a perfect square for all integers $n \geq 0$. Find a and b.

5. Show that if $m, n \geq 0$ are integers, then

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer.

6. For $s > 1$, the Riemann zeta function $\zeta(s)$ is defined by the convergent infinite series

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Find the exact value of the sum

$$\sum_{n=2}^{\infty} (-1)^n \frac{n-1}{n(n+1)} \zeta(n).$$

7. Suppose that $a_1 a_2 \cdots a_r$ is any string of letters taken from the alphabet $\{1, \ldots, n\}$ (repeats allowed). Show that it is possible to transform this string into a string $b_1 \cdots b_s$, where $b_1 < \cdots < b_s$, by a succession of moves of the following type:

(i) cyclically permute the string: $a_1 \cdots a_{r-1} a_r \to a_r a_1 \cdots a_{r-1}$;

(ii) replace any two adjacent equal letters by just one of the same letter;

(iii) replace two successive letters $i\, j$ in the string by $j\, i$ *provided* $|i - j| > 1$;

(iv) replace three successive letters of the form $i\,(i+1)\,i$ by $(i+1)\,i\,(i+1)$, or vice versa.

For example, here are some legal moves (with $n = 4$):

$$21312431 \to 23112431 \to 2312431 \to 1231243 \to 3123124 \to 1323124 \to 1232124$$

8. Suppose that A and B are non-empty subsets of the set $\mathbb{F}_p = \{0, 1, \ldots, p-1\}$, where p is a prime. Form $A + B$, the set of elements of $\mathbb{F}_p$ of the form $a + b$, where $a \in A$, $b \in B$, and the addition is taken modulo p. Show that

$$|A + B| \geq \min\{p, |A| + |B| - 1\},$$

where $|S|$ denotes the number of elements in a set S.

9. Let $\alpha = q - 1/q$. Show that for $k \in \mathbb{N}$,

$$\frac{q^k - (-q^{-1})^k}{q + q^{-1}}$$

is a polynomial in α with positive integer coefficients.

10. Let X be a nonempty set with a multiplication of elements, denoted by juxtaposition. Assume that $(xy)z = x(yz)$ for all $x, y, z \in X$, so that in any expression brackets may be dispensed with. Suppose further that for each $x \in X$ there exists a unique element $x' \in X$ such that

$$x = xx'x \quad \text{and} \quad x'xx' = x'.$$

Prove that $(xy)' = y'x'$ for all $x, y \in X$.