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1. Imagine an analogue watch with the usual hour hand, minute hand, and second hand. At how many times each day are two of the hands pointing in exactly opposite directions?

Solution. To clarify the question, the tacit assumptions are that the hands all begin at the 12 o'clock position at midnight, and move continuously in a clockwise direction at constant speeds (not ticking in discrete units, as real hands might). If we measure the direction of a hand by the clockwise angle from 12 o'clock (in radians), and time in fractions of a day after midnight, then the direction of the hour hand at time t is $4\pi t$ (two full revolutions every day), the direction of the minute hand is $48\pi t$ (24 full revolutions every day), and the direction of the second hand is $2880\pi t$ ($60 \times 24 = 1440$ full revolutions each day). To consider a single day, we restrict t to the semi-closed interval $[0, 1)$.

Now since the hour hand and minute hand start and finish the day together, and the minute hand makes 22 more revolutions, there must be 22 times each day at which the hour hand and minute hand point in opposite directions. Similarly, there must be 1438 times each day at which the second hand and minute hand point in opposite directions, and 1416 times each day at which the second hand and hour hand point in opposite directions. But we cannot simply conclude that the answer is $22 + 1438 + 1416$, because that would overcount any times at which two of the hands were pointing in the same direction and the other hand was opposite.

To find these special times, note that two hands are pointing in opposite directions if and only if the difference between their directions is $\pi + 2k\pi$ for some integer k. So the hour hand and minute hand are pointing in opposite directions exactly when $44\pi t = \pi + 2k\pi$ for some integer k, which means that t is in the following set of times:

$$A = \left\{ \frac{2k+1}{44} \,\middle|\, k \in \mathbb{Z},\ 0 \leq k \leq 21 \right\}.$$

Similarly, the hour hand and the second hand are pointing in opposite directions at the following times:

$$B = \left\{ \frac{2\ell+1}{2876} \,\middle|\, \ell \in \mathbb{Z},\ 0 \leq \ell \leq 1437 \right\},$$

and the minute hand and second hand are pointing in opposite directions at the following times:

$$C = \left\{ \frac{2m+1}{2832} \,\middle|\, m \in \mathbb{Z},\ 0 \leq m \leq 1415 \right\}.$$

Now we need to determine the intersections of these sets. We have

$$A \cap B = \left\{ \frac{1}{4}, \frac{3}{4} \right\} \quad \text{(i.e. 6am and 6pm)},$$

because $\frac{2k+1}{44} = \frac{2\ell+1}{2876}$ simplifies to $719(2k+1) = 11(2\ell+1)$, which forces 11 to divide $2k+1$, which only happens (in the stipulated range of values of k) when $k = 5$ or $k = 16$. By contrast,

$$A \cap C = B \cap C = \emptyset,$$

because $\frac{2k+1}{44} = \frac{2m+1}{2832}$ simplifies to $708(2k+1) = 11(2m+1)$, which is impossible as the left-hand side is even and the right-hand side is odd, and $\frac{2\ell+1}{2876} = \frac{2m+1}{2832}$ simplifies to $708(2\ell+1) = 719(2m+1)$, which is impossible for the same reason. So the answer is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B \cap C| = 22 + 1438 + 1416 - 2 = 2874.$$

2. A bee wants to fly on the real line from the point 0 to the point 1, visiting n flowers which are positioned at the points $\frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1}$ (here n is some fixed positive integer). The bee chooses at random, with equal probabilities, one of the $n!$ possible orderings of the flowers. It flies from 0 to the first flower, from there to the second flower, and so on through all the flowers in the chosen order, before flying on to 1. What is the expected total distance it will fly?

Solution. Let X_0 be the distance from 0 to the first flower visited, X_i ($i = 1, \cdots, n-1$) the distance from the ith flower to the $(i+1)$th flower, and X_n the distance from the last flower visited to 1. We need to evaluate the expectation $E(X_0 + X_1 + \cdots + X_n)$, and the simple key observation is that it equals $E(X_0) + E(X_1) + \cdots + E(X_n)$. Each of the n flowers is equally likely to be the first one visited, so $E(X_0)$ is the average distance of the flowers from 0, which is $\frac{1}{2}$. Similarly, $E(X_n)$ is the average distance of the flowers from 1, which is also $\frac{1}{2}$.

For any $i \in \{1, 2, \cdots, n-1\}$, each of the $\binom{n}{2}$ (unordered) pairs of flowers is equally likely to be the pair consisting of the ith and $(i+1)$th, so $E(X_i)$ is the average distance between a pair of distinct flowers. Here is a way of finding this average distance without computation. The choice of a pair of distinct flowers F and F', with $F < F'$, divides the interval $[0, 1]$ into the three intervals $[0, F]$, $[F, F']$ and $[F', 1]$, the sum of whose lengths is 1. If we imagine the interval $[0, 1]$ closed up into a circle (identify 1 with 0), these are three arcs whose endpoints are drawn from an evenly distributed set of $n + 1$ points on the circumference. By symmetry, the expected length of all three arcs is the same, hence $E(X_i) = \frac{1}{3}$.

Thus the answer to the question is $\frac{1}{2} + (n-1)\frac{1}{3} + \frac{1}{2} = \frac{n+2}{3}$.

3. The sisters Alice and Bess want to practise their arithmetic, so their father invents the following game. He begins by choosing a composite number n_0 which is at least 6. Alice and Bess then take turns saying numbers $n_1, n_2, n_3, \cdots$ (with Alice saying n_1, Bess saying n_2, Alice saying n_3, and so on) in such a way that at each step the new number n_i is the sum of two integers ≥ 2 of which n_{i-1} is the product. The winner is the first player to say a prime number. For example, if $n_0 = 16$, then Alice can say either 8 or 10, because $8 = 4 + 4$ and $10 = 2 + 8$. Saying 10 would be a bad move, because Bess would then win by saying 7 (because $7 = 2 + 5$). So Alice should say 8, which forces Bess to say 6, allowing Alice to say 5 and win. Prove that there are infinitely many starting numbers n_0 for which Bess is guaranteed to win if she plays correctly, no matter what Alice does.

Solution. Note that there are three famous conjectures which, if true, would imply this easily. The Twin Prime Conjecture states that there are infinitely many primes p such that $p + 2$ is prime: for such p, if $n_0 = p^2$ then Alice is forced to say $2p$ and Bess can then win by saying $p+2$. The Sophie Germain Prime Conjecture states that there are infinitely many primes p such that $2p + 1$ is prime: for such p, if $n_0 = p^3$ then Alice is forced to say $p + p^2 = p(p + 1)$ and

Bess can then win by saying $2p+1 = p+(p+1)$. Finally, the Goldbach Conjecture states that every even integer ≥ 4 is the sum of two primes. If this is true, there must be an infinite number of triples of primes (p, q, r) such that $q + r = 2p - 4$. For such a triple, if $n_0 = qr$ then Alice is forced to say $q + r = 2(p - 2)$, and Bess can then win by saying p.

In the absence of such results, one can argue as follows. It is clear that, given $n_0 \geq 6$, the sequence $n_0, n_1, n_2, \cdots$ is strictly decreasing and bounded below by 5; in particular, the game must end in finite time. By a basic principle of game theory, it follows that for every n_0, either Alice has a guaranteed winning strategy or Bess has one. To prove this fact for this particular game, we can use induction on n_0, as follows. If $n_0 = 6$, it is obvious that $n_1 = 5$ and Alice wins. Suppose that $n_0 > 6$ and the claim is known for smaller values of n_0. The possible values of n_1 which Alice has to consider are $d + n_0/d$, where d ranges over divisors of n_0 such that $1 < d < n_0$. If any of these numbers is prime, Alice can win straight away. If these numbers are all composite, then they can be thought of as starting numbers n_0 for different runs of the game, with the roles of the players reversed so that Bess now moves first. By the induction hypothesis, each choice comes with a guaranteed winning strategy for either Alice or Bess. If any of the choices results in Alice being guaranteed to win, she can win by choosing that; if none of them does, then Bess is guaranteed to win.

Now suppose, for a contradiction, that there are only finitely many numbers n_0 for which Bess has a guaranteed winning strategy. Then there must be a largest such number, say N. But then consider the case $n_0 = p^2$, where p is a prime greater than $N/2$: we must have $n_1 = 2p$. Since $2p$ is a composite number greater than N, our assumption means that if $2p$ were chosen as n_0, Alice would have a guaranteed winning strategy. So when Alice is forced to choose it as n_1, Bess must have a guaranteed winning strategy. So $n_0 = p^2$ gives Bess a guaranteed winning strategy, contradicting our assumption that N was the largest such composite number.

4. Let n be an odd integer ≥ 3, and let $x_1, x_2, \cdots, x_n$ be any real numbers, not necessarily positive. Prove that

$$(n-1)\max\{x_1^2, x_2^2, \cdots, x_n^2\} \;+\; (x_1 + x_2 + \cdots + x_n)^2 \;\geq\; x_1^2 + x_2^2 + \cdots + x_n^2.$$

Solution. If all $x_i = 0$, we clearly have equality. Moreover, if we multiply all x_i by a positive real number λ, both sides of the inequality are multiplied by λ^2. So we may assume that $\max\{x_1^2, x_2^2, \cdots, x_n^2\} = 1$, in which case we must prove that $\sum_{i<j} x_i x_j \geq \frac{1-n}{2}$. We will in fact show that this inequality holds for all $(x_1, x_2, \cdots, x_n)$ in the n-dimensional cube defined by the inequalities $-1 \leq x_i \leq 1$. Let $f(x_1, \cdots, x_n)$ denote $\sum_{i<j} x_i x_j$. Since the cube is compact (a closed and bounded region in $\mathbb{R}^n$) and f is continuous, there is definitely a point $(a_1, a_2, \cdots, a_n)$ in the cube where the minimum value of f is attained, and it suffices to show that $f(a_1, \cdots, a_n) \geq \frac{1-n}{2}$. This question falls under the scope of various standard optimization techniques, but we will give an elementary explanation.

We first observe that $(a_1, a_2, \cdots, a_n)$ is either a vertex of the cube (i.e. all its coordinates equal ± 1) or lies on an edge of the cube (i.e. all but one of its coordinates equal ± 1). For otherwise, we could permute the coordinates (which clearly leaves f unchanged) to ensure that $-1 < a_1 \leq a_2 < 1$, and then it is easy to see that

$$f(a_1, a_2, a_3, \cdots, a_n) > f(a_1 - \epsilon, a_2 + \epsilon, a_3, \cdots, a_n)$$

for all $\epsilon > 0$, contradicting the supposed minimality (because for sufficiently small ϵ, the point $(a_1 - \epsilon, a_2 + \epsilon, a_3, \cdots, a_n)$ still lies in the cube).

Suppose that $(a_1, a_2, \cdots, a_n)$ is a vertex of the cube. If k of its co-ordinates are 1 and $n-k$ are -1, then clearly

$$f(a_1, \cdots, a_n) = \binom{k}{2} + \binom{n-k}{2} - k(n-k) = \frac{(2k-n)^2 - n}{2} \geq \frac{1-n}{2},$$

as required. (Here we have used the fact that n is odd.)

The only remaining possibility is that $(a_1, a_2, \cdots, a_n)$ lies on an edge but is not a vertex. Permuting coordinates, we can assume that $a_1 \in (-1, 1)$, $a_2 = a_3 = \cdots = a_{k+1} = 1$, and $a_{k+2} = a_{k+3} = \cdots = a_n = -1$ for some $0 \leq k \leq n-1$. We have

$$\begin{aligned} f(x, a_2, \cdots, a_n) &= x(k - (n-1-k)) + \binom{k}{2} + \binom{n-1-k}{2} - k(n-1-k) \\ &= (2k+1-n)x + \frac{(2k+1-n)^2 + 1 - n}{2}, \end{aligned}$$

and our assumption means that the minimum value of this constant or linear function of x on the interval $[-1, 1]$ is attained at the interior point a_1. Clearly this can only happen if the function is constant, so we must have $k = \frac{n-1}{2}$, which means that $f(x, a_2, \cdots, a_n) = \frac{1-n}{2}$. Again we have the required inequality.

5. Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ be unit vectors in $\mathbb{R}^3$: that is, $|\mathbf{u}_i| = 1$ for all i, where $|\mathbf{w}|$ denotes the length of the vector $\mathbf{w}$. Assume that $|\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n| > n-2$, and that $\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \cdots + \lambda_n\mathbf{u}_n = 0$ for some nonnegative real numbers λ_i. Prove that $\lambda_i = 0$ for all i.

Solution. (Actually $\mathbb{R}^3$ could be replaced here by $\mathbb{R}^d$ for any d.) Suppose for a contradiction that some λ_i is nonzero. By renumbering the vectors if necessary, we can assume that $\lambda_n = \max\{\lambda_i\} > 0$. Then using the assumption that $|\sum_{i=1}^n \mathbf{u}_i| > n-2$ and the triangle inequality, we deduce

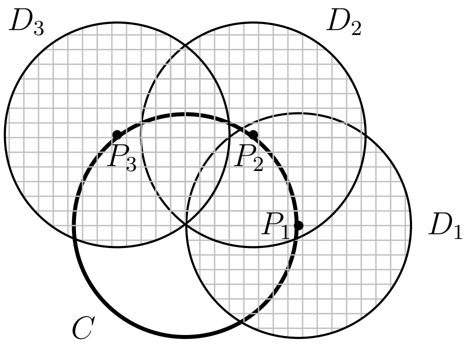
$$\begin{aligned} (n-2)\lambda_n < \lambda_n \left|\sum_{i=1}^n \mathbf{u}_i\right| &= \left|\sum_{i=1}^n \lambda_n \mathbf{u}_i\right| \\ &= \left|\sum_{i=1}^{n-1} (\lambda_n - \lambda_i)\mathbf{u}_i\right| \qquad \text{(since } \sum_{i=1}^n \lambda_i \mathbf{u}_i = 0\text{)} \\ &\leq \sum_{i=1}^{n-1} |(\lambda_n - \lambda_i)\mathbf{u}_i| = \sum_{i=1}^{n-1} (\lambda_n - \lambda_i), \end{aligned}$$

which can be rearranged to give $\sum_{i=1}^{n-1} \lambda_i < \lambda_n$. But the triangle inequality also implies

$$\begin{aligned} \lambda_n &= |\lambda_n \mathbf{u}_n| \\ &= \left|\sum_{i=1}^{n-1} -\lambda_i \mathbf{u}_i\right| \qquad \text{(since } \sum_{i=1}^n \lambda_i \mathbf{u}_i = 0\text{)} \\ &\leq \sum_{i=1}^{n-1} |-\lambda_i \mathbf{u}_i| = \sum_{i=1}^{n-1} \lambda_i, \end{aligned}$$

so we have the required contradiction.

6. Fix a positive integer $n \geq 3$. Let $P_1, P_2, \cdots, P_n$ be points which lie on a circle C of radius 1, and let D_i denote the disc with centre P_i and radius 1. A possible picture when $n = 3$ is:



Find the maximum possible area of the union $D_1 \cup D_2 \cup \cdots \cup D_n$.

Solution. Note that the maximum possible area must exist, because the area is clearly a continuous function of $P_1, P_2, \cdots, P_n$, each of which ranges over a closed and bounded set.

To find the maximum, we can assume that $P_1, P_2, \cdots, P_n$ are distinct and numbered in consecutive anti-clockwise order. For ease of notation, set $P_{n+1} = P_1$, $D_{n+1} = D_1$. Let O denote the centre of C; note that O is on the boundary of all the discs D_i. For any point S, $S \in D_i$ if and only if P_i lies in the intersection of C with the disc of radius 1 and centre S, which is some arc of the circle C. So either $S \in D_1 \cap \cdots \cap D_n$, or there is a unique $i(S)$ such that $S \in D_{i(S)}$, $S \notin D_{i(S)+1}$.

Now $D_1 \cap \cdots \cap D_n$ contains points other than O if and only if $P_1, \cdots, P_n$ all lie on one side of some line through O; this can be seen from the fact that apart from O, all other points of D_i lie on the same side as P_i of the line through O perpendicular to OP_i. So we have two cases.

Case 1: $P_1, \cdots, P_n$ do not all lie on one side of (or on) any line through O. Then $D_1 \cap \cdots \cap D_n = \{O\}$, and the union $D_1 \cup \cdots \cup D_n$ can be written as the disjoint union

$$(D_1 \setminus (D_1 \cap D_2)) \amalg (D_2 \setminus (D_2 \cap D_3)) \amalg \cdots \amalg (D_n \setminus (D_n \cap D_1)) \amalg \{O\}.$$

Case 2: $P_1, \cdots, P_n$ all lie on one side of (or on) some line through O. We renumber so that P_1 and P_n are the outermost points; then $D_n \cap D_1$ is the same as $D_1 \cap \cdots \cap D_n$, so the union $D_1 \cup \cdots \cup D_n$ can be written as the disjoint union

$$(D_1 \setminus (D_1 \cap D_2)) \amalg (D_2 \setminus (D_2 \cap D_3)) \amalg \cdots \amalg (D_{n-1} \setminus (D_{n-1} \cap D_n)) \amalg D_n.$$

We must now calculate the area of $D_i \setminus (D_i \cap D_{i+1})$. Let θ_i be the angle $\angle P_i O P_{i+1}$. Let Q_i denote the other point which is at distance 1 from both P_i and P_{i+1}, and let R_i be the midpoint of the line segment $P_i P_{i+1}$, which is also the midpoint of OQ_i. Clearly the angle $\angle Q_i P_i O = \pi - \theta_i$, so the area of the sector of D_i bounded by $Q_i P_i$ and OP_i is $\frac{\pi - \theta_i}{2}$. Also the distance OR_i is $2\cos\frac{\theta_i}{2}$ and the distance $R_i P_i$ is $\sin\frac{\theta_i}{2}$, so the area of the triangle $Q_i P_i O$ is $\frac{\sin\theta_i}{2}$. Hence the area of the segment of D_i bounded by the chord OQ_i is $\frac{\pi - \theta_i - \sin\theta_i}{2}$. This is half of the overlap $D_i \cap D_{i+1}$, whose area is therefore $\pi - \theta_i - \sin\theta_i$, and the area of $D_i \setminus (D_i \cap D_{i+1})$ is $\theta_i + \sin\theta_i$.

In Case 1, we have $0 < \theta_i < \pi$ for $i = 1, 2, \cdots, n$, and $\sum_{i=1}^n \theta_i = 2\pi$. By the above disjoint union, the area of $D_1 \cup \cdots \cup D_n$ is:

$$\sum_{i=1}^n (\theta_i + \sin\theta_i) = 2\pi + \sum_{i=1}^n \sin\theta_i.$$

By basic methods in constrained optimization (e.g. Lagrange multipliers), the maximum value of $\sum_{i=1}^n \sin\theta_i$ subject to the constraints on θ is $n \sin\frac{2\pi}{n}$, occurring when all $\theta_i = \frac{2\pi}{n}$. So the maximum area in this case occurs when the points P_i are equally spaced around the circle C, and it is $2\pi + n\sin\frac{2\pi}{n}$.

In Case 2, we have $0 < \theta_i < \pi$ for $i = 1, 2, \cdots, n-1$, and $\sum_{i=1}^{n-1} \theta_i \le \pi$. By the above disjoint union, the area of $D_1 \cup \cdots \cup D_n$ is:

$$\pi + \sum_{i=1}^{n-1} (\theta_i + \sin\theta_i).$$

For any fixed value of $\sum_{i=1}^{n-1} \theta_i$, say k, the maximum value of $\sum_{i=1}^{n-1} \sin\theta_i$ is $(n-1)\sin\frac{k}{n-1}$, occurring when $\theta_i = \frac{k}{n-1}$ for all $i = 1, 2, \cdots, n-1$. Moreover, since $\sin$ is increasing on $[0, \frac{\pi}{2}]$, we have

$$\pi + k + (n-1)\sin\frac{k}{n-1} \le \pi + \pi + (n-1)\sin\frac{\pi}{n-1}.$$

So the maximum area in this case occurs when P_1 and P_n are antipodal and the other points P_i are equally spaced around one of the semicircles between them, and it is $2\pi + (n-1)\sin\frac{\pi}{n-1}$. It is easy to see that this is less than the maximum in Case 1: if $n = 3$ the values can be computed exactly, and for $n \ge 4$ we have $\sin\frac{2\pi}{n} > \sin\frac{\pi}{n-1} > 0$. So the overall maximum is $2\pi + n\sin\frac{2\pi}{n}$.

7. For real numbers a, b, c, d with $a \ne 0$, consider the equation $\overline{z} = az^3 + bz^2 + cz + d$, where the unknown z is a complex number and $\overline{z}$ denotes the conjugate of z. What are the minimum and maximum number of solutions this equation can have, for different choices of a, b, c, d?

Solution. We will show that the equation $\overline{z} = az^3 + bz^2 + cz + d$ always has between three and seven solutions. We can see that these extremes are attained as follows:

- The equation $\overline{z} = z^3 + z$ has solutions $0, \pm\sqrt{2}i$. These are the only solutions on the imaginary axis, because when $z = iy$, the equation becomes $y^3 - 2y = 0$. If z were a solution not on the imaginary axis, then $z^3 = \overline{z} - z$ would be purely imaginary, so the principal argument of z would have to be either $\pm\frac{\pi}{6}$ or $\pm\frac{5\pi}{6}$. But for each of these arguments, it is easy to see that z^3 lies on the opposite half of the imaginary axis to $\overline{z} - z$. So there are exactly three solutions.
- The equation $\overline{z} = -z^3 + \frac{5}{4}z$ has seven solutions, namely

  $$0,\ \pm\frac{1}{2},\ \pm\frac{\sqrt{13}+\sqrt{3}i}{4},\ \text{and}\ \pm\frac{\sqrt{13}-\sqrt{3}i}{4}.$$

  The verification that these are solutions is routine; that there are no more solutions is a special case of Case 2 below.

We must now show that the number of solutions is always between three and seven. We will use the known fact (easily derived from basic calculus) that for any real numbers p, q, the cubic $x^3 + px + q$ has one real root if $4p^3 + 27q^2 > 0$, two real roots if $4p^3 + 27q^2 = 0$ (unless $p = q = 0$, in which case 0 is the unique root), and three real roots if $4p^3 + 27q^2 < 0$.

By replacing the variable z with $z - \frac{b}{3a}$, we can remove the z^2 term, i.e. we may assume that $b = 0$. Moreover, by replacing the variable z with $\frac{z}{\sqrt{|a|}}$, we can scale the coefficient of z^3 to either 1 or -1, i.e. we may assume that $a = \pm 1$.

Case 1: $a = 1$, so the equation is $\overline{z} = z^3 + cz + d$. The real solutions of this equation are the roots of the cubic $f(x) = x^3 + (c-1)x + d$. To find non-real roots, we write $z = x + iy$ and equate real and imaginary parts, cancelling a common factor of y from the imaginary-parts equation ($y \neq 0$ because z is to be non-real). This results in two equations for x and y:

$$x^3 - 3xy^2 + (c-1)x + d = 0,$$
$$3x^2 - y^2 + (c+1) = 0.$$

From the second equation we have $y^2 = 3x^2 + (c+1)$, so x is a root of the cubic $g(x) = x^3 + \frac{c+2}{4}x - \frac{d}{8}$ and must satisfy $x^2 > -\frac{c+1}{3}$. For any such x we have two possible values of y (reflecting the fact that z is a solution if and only if $\overline{z}$ is). To sum up, the number of solutions of $\overline{z} = z^3 + cz + d$ equals $A + 2B$, where A is the number of real roots of $f(x)$, and B is the number of real roots of $g(x)$ which satisfy $x^2 > -\frac{c+1}{3}$. We must show that $3 \leq A + 2B \leq 7$.

The only way we could have $A + 2B < 3$ is if $A \leq 2$ and $B = 0$. To say that $B = 0$ is to say that $c \leq -1$ and all the real roots of $g(x)$ lie in the closed interval $[-\sqrt{-\frac{c+1}{3}}, \sqrt{-\frac{c+1}{3}}]$. This implies that $g(-\sqrt{-\frac{c+1}{3}}) \leq 0$ and $g(\sqrt{-\frac{c+1}{3}}) \geq 0$, which boil down to the inequalities:

$$-\frac{2-c}{12}\sqrt{-\frac{c+1}{3}} \leq \frac{d}{8} \leq \frac{2-c}{12}\sqrt{-\frac{c+1}{3}}.$$

From this we deduce that $27d^2 \leq -4(c-2)^2(c+1)$, so

$$4(c-1)^3 + 27d^2 \leq 12c - 20 \leq -32,$$

which would imply that $A = 3$, contradicting our assumption. So $A + 2B \geq 3$ is proved.

The only way we could have $A + 2B > 7$ is if $A \geq 2$ and $B = 3$. The fact that $g(x)$ has three real roots implies that $(c+2)^3 + \frac{27d^2}{4} < 0$, so in particular $c < -2$; we are moreover assuming that all of these roots satisfy $x^2 > -\frac{c+1}{3}$, so there are no roots of $g(x)$ in the closed interval $[-\sqrt{-\frac{c+1}{3}}, \sqrt{-\frac{c+1}{3}}]$. But on the other hand, there must be a root of $g(x)$ between the two critical points $\pm\sqrt{-\frac{c+2}{12}}$, which lie in this interval. This contradiction shows that $A + 2B \leq 7$, concluding our analysis of Case 1.

Case 2: $a = -1$, so the equation is $\overline{z} = -z^3 + cz + d$. Proceeding as in Case 1, we see that the number of solutions is $A + 2B$ where A is the number of real roots of $f(x) = x^3 - (c-1)x - d$, and B is the number of real roots of $g(x) = x^3 - \frac{c+2}{4}x + \frac{d}{8}$ which satisfy $x^2 > \frac{c+1}{3}$.

As in Case 1, $A + 2B < 3$ would imply $B = 0$, so $c \geq -1$ and all the real roots of $g(x)$ lie in the closed interval $[-\sqrt{\frac{c+1}{3}}, \sqrt{\frac{c+1}{3}}]$. This implies that $g(-\sqrt{\frac{c+1}{3}}) \leq 0$ and $g(\sqrt{\frac{c+1}{3}}) \geq 0$, which boil down to the inequalities:

$$-\frac{c-2}{12}\sqrt{\frac{c+1}{3}} \leq -\frac{d}{8} \leq \frac{c-2}{12}\sqrt{\frac{c+1}{3}}.$$

Hence we have $c \geq 2$ and $27d^2 \leq 4(c-2)^2(c+1)$, so

$$-4(c-1)^3 + 27d^2 \leq -12c + 20 \leq -4,$$

which implies that $A = 3$, contradicting our assumption.

Finally, $A + 2B > 7$ would imply $A \geq 2$ and $B = 3$. The fact that $f(x)$ has more than one real root implies that $-4(c-1)^3 + 27d^2 \leq 0$, so in particular $c \geq 1$. The fact that all the roots of $g(x)$ satisfy $x^2 > \frac{c+1}{3}$ implies that there are no roots in the closed interval $[-\sqrt{\frac{c+1}{3}}, \sqrt{\frac{c+1}{3}}]$. But there must be a root of $g(x)$ between the two critical points $\pm\sqrt{\frac{c+2}{12}}$, which lie in this interval. This gives the required contradiction.

If instead of cubic polynomials we had considered polynomials of degree n, where $n > 1$, the minimum number of solutions would have been n and the maximum would have been $3n-2$ (see L. Geyer, 'Sharp bounds for the valence of certain harmonic polynomials', Proc. Amer. Math. Soc. 136 (2008), no. 2, 549–555). Allowing the coefficients of the polynomial to be complex would not have made any difference.

8. A famous theorem in algebra says that any $n \times n$ integer matrix A can be written as a matrix product XDY, where X and Y are integer matrices with determinant ± 1, and D is a diagonal matrix with nonnegative integer diagonal entries $d_1, d_2, \cdots, d_n$ such that d_{i+1} is a multiple of d_i for all $1 \le i \le n-1$. The numbers $d_1, d_2, \cdots, d_n$ are uniquely determined and are called the *invariant factors* of A. Find the invariant factors of the matrix $A = (a_{ij})_{i,j=1}^n$, where $a_{ij} = i^j$.

Solution. We have the identity

$$i^j = \sum_{k \le i,j} \binom{i}{k} k!\, S(j,k),$$

where $S(j,k)$ denotes the Stirling number of the second kind, i.e. the number of ways of partitioning a set with j elements into k blocks. The proof is that the left-hand side counts all functions $f : \{1, \cdots, j\} \to \{1, \cdots, i\}$, and the number of functions whose image is a fixed subset K is $|K|!\, S(j, |K|)$.

Hence $A = XDY$ where X is the lower-triangular matrix whose (i,k)-entry is $\binom{i}{k}$, D is the diagonal matrix whose (k,k)-entry is $k!$, and Y is the upper-triangular matrix whose (k,j)-entry is $S(j,k)$. Since X and Y have all diagonal entries 1, $\det(X) = \det(Y) = 1$. Moreover, it is clear that $(i+1)!$ is a multiple of $i!$ for all $1 \le i \le n-1$, so the invariant factors of A are $1!, 2!, 3!, \cdots, n!$.

9. In the complex vector space $\mathbb{C}^2$ we define an inner product by

$$(z_1, z_2) \cdot (w_1, w_2) = z_1\overline{w_1} + z_2\overline{w_2}, \text{ for all } z_1, z_2, w_1, w_2 \in \mathbb{C}.$$

An element $(z_1, z_2) \in \mathbb{C}^2$ is a *unit vector* if $(z_1, z_2) \cdot (z_1, z_2) = 1$. Show that it is impossible to have five unit vectors $(z_1^{(a)}, z_2^{(a)})$, $a = 1, 2, 3, 4, 5$, no two of which are scalar multiples of each other, such that $|(z_1^{(a)}, z_2^{(a)}) \cdot (z_1^{(b)}, z_2^{(b)})|$ is the same for all pairs (a, b) with $a \ne b$.

Solution. Assume for a contradiction that we have five non-proportional unit vectors $(z_1^{(a)}, z_2^{(a)})$, $a = 1, 2, 3, 4, 5$, satisfying the condition in the question; let c denote the common value of $|(z_1^{(a)}, z_2^{(a)}) \cdot (z_1^{(b)}, z_2^{(b)})|$ for all $a \ne b$. Clearly c is a nonnegative real number, and $c \le 1$ by the Cauchy-Schwarz Inequality. Since the inner product is a sesquilinear form, we can multiply each unit vector by a scalar (complex number) of modulus 1 without affecting anything.

Let SU_2 denote the group of 2×2 complex matrices of the form $\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$, where $|\alpha|^2 + |\beta|^2 = 1$. For any such matrix, we have

$$\begin{aligned}
&(z_1, z_2)\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) \cdot (w_1, w_2)\left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right) = (\alpha z_1 - \overline{\beta} z_2, \beta z_1 + \overline{\alpha} z_2) \cdot (\alpha w_1 - \overline{\beta} w_2, \beta w_1 + \overline{\alpha} w_2) \\
&= (|\alpha|^2 z_1\overline{w_1} - \alpha\beta z_1\overline{w_2} - \overline{\alpha\beta} z_2\overline{w_1} + |\beta|^2 z_2\overline{w_2}) + (|\beta|^2 z_1\overline{w_1} + \alpha\beta z_1\overline{w_2} + \overline{\alpha\beta} z_2\overline{w_1} + |\alpha|^2 z_2\overline{w_2}) \\
&= z_1\overline{w_1} + z_2\overline{w_2} \\
&= (x_1, x_2) \cdot (y_1, y_2).
\end{aligned}$$

So multiplication by an element of SU_2 preserves the property of being a unit vector, and would preserve the supposed property of the 5 unit vectors we are investigating. Moreover, SU_2 acts transitively on the set of unit vectors, because the orbit of $(1,0)$ consists of all the top rows of matrices in SU_2, i.e. all unit vectors (α,β). So we may assume that the last of our five unit vectors is $(1,0)$. Hence we have $|z_1^{(a)}| = c$ for $a = 1,2,3,4$. As noted above, we can multiply each unit vector by a complex number of modulus 1, so we can assume that $z_1^{(a)} = c$ for $a = 1,2,3,4$. Hence $|z_2^{(a)}| = \sqrt{1-c^2}$ for $a = 1,2,3,4$. Since the vectors are distinct, we cannot have $c = 1$, so we must have $c < 1$.

At this stage the remaining equations are that

$$|c^2 + z_2^{(a)}\overline{z_2^{(b)}}| = c, \text{ for all } 1 \leq a \neq b \leq 4.$$

If we set $z_2^{(a)} = \sqrt{1-c^2}\, e^{i\theta_a}$ for $a = 1,2,3,4$, where without loss of generality $0 \leq \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$, then these equations become

$$(c^2 + (1-c^2)e^{i(\theta_a-\theta_b)})(c^2 + (1-c^2)e^{i(\theta_b-\theta_a)}) = c^2, \text{ for all } 1 \leq a \neq b \leq 4,$$

which simplifies to

$$\cos(\theta_a - \theta_b) = 1 - \frac{1}{2c^2}, \text{ for all } 1 \leq a \neq b \leq 4.$$

The case $c = 0$ gets ruled out along the way in this simplification, and we now see that in fact $c \geq \frac{1}{2}$. Let $x = \cos^{-1}(1 - \frac{1}{2c^2})$, so $\pi/3 < x \leq \pi$. Then for all $1 \leq b < a \leq 4$, we know that $\theta_a - \theta_b$ is either x or $2\pi - x$. Thus the equation $(\theta_2 - \theta_1) + (\theta_3 - \theta_2) = \theta_3 - \theta_1$ must be either $x + x = 2\pi - x$ or $(2\pi - x) + (2\pi - x) = x$, and the second leads to $x = 4\pi$ which is impossible, so $x = 2\pi/3$ and $\theta_2 - \theta_1 = \theta_3 - \theta_2 = 2\pi/3$. Similarly we find that $\theta_4 - \theta_3 = 2\pi/3$, but then $\theta_4 - \theta_1 = 2\pi$ is neither x nor $2\pi - x$, which gives the desired contradiction. The proof is finished.

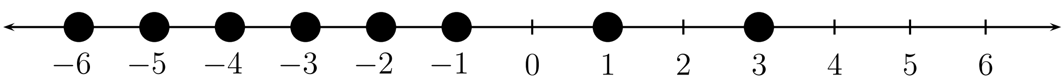
Incidentally, this argument has produced four unit vectors which do have the required property, namely

$$(1,0),\ (\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}),\ (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}i),\ (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}}i).$$

These form the vertices of a regular tetrahedron. The maximum number of such "equiangular" unit vectors in $\mathbb{C}^d$ for general d is unknown.

10. Imagine placing infinitely many identical coins at integer points on the real line (at most one coin at each integer). Call such a placement *allowable* if, for all sufficiently large positive integers N, there is a coin at $-N$ but not at N. Thus every allowable placement has a contiguous block of coins on the left, and there is some integer a (the "first gap") which is minimal among those where there is no coin. Call an allowable placement *well-spaced* if there are no two coins at positions $b, b+1$ where $b > a$ (i.e. no adjacent coins to the right of the first gap). By a *move* from one allowable placement to another, we mean a move of a single coin two places to the right, i.e. removing a coin at i and replacing it at the previously empty position $i+2$ for some i.

For any integers m and n, define an allowable placement $P_{m,n}$ in which the coins are placed at the odd integers $\leq 2m-1$ and the even integers $\leq 2n$. Here is a picture of $P_{2,-1}$:



Let $f(m,n,w)$ be the number of well-spaced placements which can be obtained from $P_{m,n}$ by a sequence of exactly w moves (the intermediate placements do not have to be well-spaced). Prove that $f(m,n,w)$ is independent of m and n.

Solution. In fact, $f(m,n,w)$ is equal to the number of partitions of w, i.e. ways to write w as a sum of positive integers (with repetition allowed, and without considering order). This is a special case of a major combinatorial result, known as Chung's Conjecture before it was proved in 1952 (see G. D. James, 'Some combinatorial results involving Young diagrams', Math. Proc. Camb. Phil. Soc. 83 (1978), 1–10).

It is clear that replacing m and n with $m+c$ and $n+c$ for any integer c would not change the problem, since it would merely shift all coins $2c$ places. So $f(m,n,w) = g(m-n,w)$ for some function g of two integer variables (the second of which is nonnegative). Similarly, shifting all coins one place to the right corresponds to replacing (m,n) with $(n+1,m)$, so $f(m,n,w) = f(n+1,m,w)$, showing that $g(d,w) = g(1-d,w)$. To complete the proof that g is independent of its first variable, it suffices to show that $g(d,w) = g(-d,w)$, or equivalently $f(m,n,w) = f(n,m,w)$. (Having proved the independence, it is easy to see that the common value of $f(m,n,w)$ for all m,n is the number of partitions of w, because in the case that $m-n$ is much larger than w, the placement resulting from the w moves is well-spaced if and only if all the moves involved coins at odd positions.)

To prove that $f(m,n,w) = f(n,m,w)$, it is enough to construct a bijection between the well-spaced placements obtained from $P_{m,n}$ by w moves and the well-spaced placements obtained from $P_{n,m}$ by w moves. It is convenient to rephrase the definition of these sets using some further notation.

To any allowable placement P we attach a triple of integers $(a(P), b(P), c(P))$ as follows. Define $a(P)$ by the rule that for all sufficiently large integers N,

$$a(P) = \#\{\text{coins at odd positions } > -2N\} - \#\{\text{coins at even positions } > -2N\}.$$

(It is clear that for large N this quantity is independent of N.) Define $b(P)$ by the requirement that if we were to move all coins in P as far left as they would go (irrespective of parity), they would occupy exactly the positions $\cdots, b(P)-2, b(P)-1, b(P)$. Finally, let $c(P)$ be the number of pairs $(i,i') \in \mathbb{Z}^2$ where $i < i'$ and P has a coin at i' but not at i.

Suppose that P' is obtained from P by a move of a single coin two places to the right as in the question. Then it is easy to see that $a(P') = a(P)$, $b(P') = b(P)$, and $c(P') = c(P) + 2$. Moreover, a short calculation gives

$$a(P_{m,n}) = m-n,\ b(P_{m,n}) = m+n,\ \text{and } c(P_{m,n}) = \binom{m-n}{2}.$$

Here $\binom{x}{2}$ means $\frac{x(x-1)}{2}$, whether x is positive or negative.

It is clear that every allowable placement P is obtained from some initial placement $P_{m,n}$ by moves as in the question. Since the a and b values are left unchanged by the moves, this initial placement is in fact uniquely determined: we must have $m = \frac{1}{2}(a(P) + b(P))$ and $n = \frac{1}{2}(b(P) - a(P))$. Moreover, the number of moves must be $\frac{1}{2}(c(P) - \binom{a(P)}{2})$ (in particular, $c(P)$ must be greater than or equal to $\binom{a(P)}{2}$ and of the same parity). So the allowable placements P obtained from $P_{m,n}$ by w moves are exactly those for which $a(P) = m-n$, $b(P) = m+n$, and $c(P) = \binom{m-n}{2} + 2w$, and those obtained from $P_{n,m}$ by w moves are those for which $a(P) = n-m$, $b(P) = m+n$, and $c(P) = \binom{n-m}{2} + 2w = m-n+\binom{m-n}{2} + 2w$.

It will therefore suffice if we can construct an involution (i.e. a self-inverse permutation) $P \mapsto \overline{P}$ of the set of well-spaced placements which has the following properties:

a) $a(\overline{P}) = -a(P)$,

b) $b(\overline{P}) = b(P)$,

c) $c(\overline{P}) = c(P) + a(P)$.

The following definition of such an involution is adapted from one given in the paper by A. Lascoux, B. Leclerc, and J.-Y. Thibon, ‘Hecke algebras at roots of unity and crystal bases of quantum affine algebras’, Commun. Math. Phys. 181 (1996), 205–263.

Let $(i_1 < \cdots < i_s)$ be the sequence of positions of all coins in P not in the contiguous block, except that we include the rightmost coin in the contiguous block if its position is odd. Since P is well-spaced, $i_j + 1 < i_{j+1}$ for all $1 \leq j \leq s-1$. Now define $\epsilon_1, \cdots, \epsilon_s$ by the rule that $\epsilon_j = 1$ if i_j is odd and $\epsilon_j = -1$ if i_j is even. Note that $\epsilon_1 + \cdots + \epsilon_s = a(P)$. If $a(P) = 0$ (i.e. there are equal numbers of 1s and (-1)s), define $\overline{P} = P$; otherwise, proceed as follows. From the sequence $1 < \cdots < s$, eliminate any consecutive pair $j < j'$ with the property that $i_j = -1$ and $i_{j'} = 1$; then eliminate such consecutive pairs from the remaining sequence, and continue in this way until what remains is a sequence $j_1 < \cdots < j_t$ such that $i_{j_1} = \cdots = i_{j_k} = 1$, $i_{j_{k+1}} = \cdots = i_{j_t} = -1$ for some $0 \leq k \leq t$. We have $k - (t-k) = a(P) \neq 0$. If $k > t-k$, we define $\overline{P}$ to be the placement obtained from P by moving the coin on $i_{j_{t-k+1}}$ to $i_{j_{t-k+1}} + 1$, the coin on $i_{j_{t-k+1}+1}$ to $i_{j_{t-k+1}+1} + 1$, and so on, up to the coin on i_{j_k} to $i_{j_k} + 1$. (Note that these are not moves of the type considered in the question.) If $k < t-k$, we define $\overline{P}$ to be the placement obtained from P by moving the coin on $i_{j_{k+1}}$ to $i_{j_{k+1}} - 1$, the coin on $i_{j_{k+2}}$ to $i_{j_{k+2}} - 1$, and so on, up to the coin on $i_{j_{t-k}}$ to $i_{j_{t-k}} - 1$. It is straightforward to check that $\overline{P}$ is well-spaced. Moreover, the two cases are inverse to each other, so we always have $\overline{\overline{P}} = P$. Thus $P \mapsto \overline{P}$ gives an involution of the set of well-spaced placements. It is easy to see from the definition that the three properties are satisfied.