The University of Sydney
School of Mathematics and Statistics
NSW 2006 Australia

SUMS Problem Competition 2006

1. For any positive real number x, let $\langle x\rangle$ denote the fractional part of x, i.e. the unique element of $[0,1)$ such that $x-\langle x\rangle$ is an integer. If N is a positive integer, the *scale* based on x and N is the set $\{0,\langle x\rangle,\langle 2x\rangle,\cdots,\langle Nx\rangle,1\}$. This has at most $N+2$ distinct elements, possibly fewer. If we list the distinct elements of the scale in order, $0=s_0<s_1<\cdots<s_k=1$, the *intervals* in the scale are the differences $s_1-s_0, s_2-s_1,\cdots,s_k-s_{k-1}$. Prove that there are at most three different intervals.

Solution. The only way there could be fewer than $N+2$ elements in the scale is if x is rational and can be written in lowest terms as $\frac{p}{q}$, with $1\le q\le N$. In this case, it is clear that the scale based on $\frac{p}{q}$ and N is $\{0,\frac{1}{q},\frac{2}{q},\cdots,\frac{q-1}{q},1\}$, and all the intervals equal $\frac{1}{q}$. In more detail: for all $0\le i\le q-1$, we have $\langle m\frac{p}{q}\rangle=\frac{i}{q}$ whenever $m\equiv ip^{-1}$ mod q, where p^{-1} denotes the multiplicative inverse of p in $\mathbb{Z}/q\mathbb{Z}$ (if $q=1$, $p^{-1}=0$). The set of values $0\le m\le N$ which satisfy this congruence is of the form

$$m_i, m_i+q, m_i+2q,\cdots,m_i+\lfloor\frac{N-m_i}{q}\rfloor q=m_i',$$

where m_i is the smallest nonnegative integer congruent to ip^{-1} mod q and m_i' is the largest integer not exceeding N satisfying the same congruence. Note that $m_i\le q-1<N$ and $m_i'\ge N-q+1$. (Of course $m_0=0$ and $m_0'=\lfloor\frac{N}{q}\rfloor q$.)

Now if x is not of the above form, we let $\frac{p}{q}$ be the largest number of the above form which is less than x, and write $x=\frac{p}{q}+\epsilon$. For each $0\le i\le q-1$, we have a set of scale values

$$\langle m_i x\rangle,\langle(m_i+q)x\rangle,\langle(m_i+2q)x\rangle,\cdots,\langle m_i'x\rangle$$

corresponding to the values which were equal to $\frac{i}{q}$ in the $\frac{p}{q}$ scale. Our claim is, firstly, that these values equal

$$\frac{i}{q}+m_i\epsilon,\frac{i}{q}+(m_i+q)\epsilon,\frac{i}{q}+(m_i+2q)\epsilon,\cdots,\frac{i}{q}+m_i'\epsilon \qquad (1)$$

respectively, and, secondly, that the scale consists exactly of the concatenation of the 'sub-scales' (1) from $i=0$ to $i=q-1$ **with no overlapping**, followed by 1. To see this, note that the claim is certainly true for ϵ sufficiently small; and as ϵ increases, the 'first time' it fails is when there is some coincidence of scale values. But such a coincidence means exactly that $\frac{p}{q}+\epsilon=\frac{p'}{q'}$ where $1\le q'\le N$, and our maximality assumption on $\frac{p}{q}$ ensures that we do not reach this point.

So the possible intervals are as follows: within each sub-scale (1), all intervals equal $q\epsilon$; and between the end of one sub-scale and (1 or) the beginning of the next, we have an interval

$$\frac{1}{q}+(m_{i+1}-m_i')\epsilon,$$

where we set $m_q = 0$ to cover the final interval also. But $m_{i+1} - m_i' \equiv p^{-1} \bmod q$, and we have the bounds

$$-N \leq m_{i+1} - m_i' \leq q - 1 - (N - q + 1) = -N + 2q - 2.$$

Hence there are at most two possible values $m_{i+1} - m_i'$ can take, and at most three possible intervals all told.

2. Find the volume of the region in $\mathbb{R}^3$ defined by the inequalities

$$|x|^{2/3} + |y|^{2/3} \leq 1, \ |x|^{2/3} + |z|^{2/3} \leq 1, \ |y|^{2/3} + |z|^{2/3} \leq 1.$$

Solution. Let R_α denote the region defined analogously but with $2/3$ replaced by a general positive exponent α. It is clear that R_α contains the cube

$$C_\alpha = \{(x, y, z) \in \mathbb{R}^3 \mid |x|, |y|, |z| \leq 2^{-1/\alpha}\}.$$

Moreover, if $(x, y, z) \in R_\alpha \setminus C_\alpha$, then exactly one of $|x|$, $|y|$, $|z|$ exceeds $2^{-1/\alpha}$. So $R_\alpha \setminus C_\alpha$ is the disjoint union of six regions congruent to

$$\{(x, y, z) \in \mathbb{R}^3 \mid 2^{-1/\alpha} < x \leq 1, \ |y|, |z| \leq (1 - x^\alpha)^{1/\alpha}\}.$$

Hence

$$\begin{aligned} \mathrm{vol}(R_\alpha) &= \mathrm{vol}(C_\alpha) + 6 \int_{2^{-1/\alpha}}^{1} 4(1 - x^\alpha)^{2/\alpha} \, \mathrm{d}x \\ &= 2^{3-3/\alpha} + \frac{24}{\alpha} \int_{1/2}^{1} u^{1/\alpha - 1}(1 - u)^{2/\alpha} \, \mathrm{d}u, \end{aligned}$$

where we have made the substitution $x = u^{1/\alpha}$ in the integral. In the case when $\alpha = 2/3$,

$$\begin{aligned} \mathrm{vol}(R_{2/3}) &= 2^{-3/2} + 36 \int_{1/2}^{1} u^{1/2}(1 - u)^3 \, \mathrm{d}u \\ &= \frac{\sqrt{2}}{4} + 36 \left[\frac{2}{3}u^{3/2} - \frac{6}{5}u^{5/2} + \frac{6}{7}u^{7/2} - \frac{2}{9}u^{9/2}\right]_{1/2}^{1} \\ &= \frac{128 - 71\sqrt{2}}{35}. \end{aligned}$$

3. Let D be a regular dodecahedron with edges of length 1. Find the shortest possible length of a path on the surface of D starting at one vertex and finishing at the antipodal vertex.

Solution. (Sketch.) It is easy to see from a picture or model that the only paths which could feasibly be minimal are of two types: one type crossing four faces and one type crossing three. We can then unfold the relevant faces and picture them as regular pentagons in the plane; the minimal length paths are now straight lines. Recall that, the edges being of length 1, the diagonals of the pentagons are of length $\tau = \frac{\sqrt{5}+1}{2}$. The first kind of path is part of a triangle whose other sides are 2τ and 1, with opposite angle $\frac{4\pi}{5}$; thus by the cosine rule its square is

$$4\tau^2 + 1 - 4\tau \cos \frac{4\pi}{5} = 6\tau + 7 \approx 16 \cdot 7.$$

The other kind of path is part of a triangle whose other sides are $\tau + 1$ and τ, with opposite angle $\frac{4\pi}{5}$; thus its square is

$$(\tau+1)^2 + \tau^2 - 2\tau(\tau+1)\cos\frac{4\pi}{5} = 7\tau + 5 \approx 16\cdot 3.$$

So the second kind of path is shorter, and the answer is $\sqrt{7\tau+5} = \sqrt{\frac{7\sqrt{5}+17}{2}}$.

4. In this problem, 'number' means positive integer. Suppose we consider two numbers to be *essentially equal* (written $\approx$) if they become the same when all zeroes are deleted from their decimal expression (for instance, $1023 \approx 120030$). For consistency with multiplication, we had better extend the notion of essential equality so that

$$a \approx b \Longleftrightarrow a \times c \approx b \times c, \text{ for any numbers } a, b, c.$$

(For instance, the fact that $2 \times 6 = 12 \approx 102 = 17 \times 6$ implies that $2 \approx 17$.) Of course, we also stipulate that $a \approx b$ and $b \approx c$ together imply $a \approx c$. Show that for any number a, there is another number b such that $a \times b \approx 1$.

Solution. Consider the numbers $1, 11, 111$, etc. Since there are only finitely many congruence classes modulo a, two of these numbers must be congruent; in other words, a has a multiple of the form $11\cdots100\cdots0$. We will show that any number of the latter form is essentially equal to 1; obviously we can forget about the string of zeroes.

We first prove by *ad hoc* methods that various other numbers are essentially equal to 1. From $15 \times 7 = 105 \approx 15$ we see that $7 \approx 1$. Then from $11 \times 13 \approx 7 \times 11 \times 13 = 1001 \approx 11$ we see that $13 \approx 1$. From $2 \approx 2 \times 7 = 14 \approx 104 = 8 \times 13$ we see that $4 \approx 1$. But also $18 \times 6 = 108 \approx 18$, so $6 \approx 1$. Thus $6 \approx 4$, so $3 \approx 2$ and $9 = 3 \times 3 \approx 2 \times 3 \approx 1$. Similarly from $10 \approx 1 \approx 4$ we get $5 \approx 2$ and $25 \approx 1$. Now $5 \times 5 = 25 \approx 205 = 5 \times 41$, so $41 \approx 5 \approx 2$; also $4 \times 23 = 92 \approx 902 = 2 \times 41 \times 11 \approx 4 \times 11$, so $23 \approx 11$. But also $23 \approx 9 \times 23 = 207 \approx 27 \approx 3 \approx 2$, so $11 \approx 2$. From $9 \approx 81 \approx 801 = 9 \times 89$ we get $89 \approx 1$, whence $2 \approx 2 \times 89 = 178 \approx 1078 = 2 \times 11 \times 49 \approx 4 \times 49 \approx 1$. This means that every number mentioned in this paragraph is essentially equal to 1.

We now note that

$$\begin{aligned} 11\cdots1 &\approx 11\cdots1 \times 2 \times 41 = 911\cdots102 \\ &\approx 911\cdots12 \times 9 = 8200\cdots08 \\ &\approx 828 = 4 \times 207 \approx 1, \end{aligned}$$

as required. It seems plausible that in fact all numbers are essentially equal to 1.

5. Let n be a positive integer. Show that the average of the numbers $(\tan\frac{\pi}{2n+1})^2, (\tan\frac{2\pi}{2n+1})^2, \cdots, (\tan\frac{n\pi}{2n+1})^2$ equals their product.

Solution. We will in fact prove an equality of polynomials:

$$(x + (\tan\frac{\pi}{2n+1})^2)(x + (\tan\frac{2\pi}{2n+1})^2)\cdots(x + (\tan\frac{n\pi}{2n+1})^2) = \sum_{j=0}^{n}\binom{2n+1}{2j}x^{n-j}. \quad (2)$$

From this equality it follows that the sum of the numbers in the question is $\binom{2n+1}{2} = n(2n+1)$ (so their average is $2n+1$), and their product is $\binom{2n+1}{2n} = 2n+1$ also. To prove (2), let $P(x)$ denote the right-hand side. Now $P(x)$ is certainly a monic polynomial of degree n, and the factors on the left-hand side are all different because $\tan$ is increasing on $(0, \frac{\pi}{2})$. So it suffices to show, for each $1 \leq k \leq n$, that $P(-(\tan \frac{k\pi}{2n+1})^2) = 0$. But if we think in terms of polynomials with complex coefficients,

$$P(-x^2) = \sum_{j=0}^{n} \binom{2n+1}{2j} (ix)^{2n-2j} = \frac{1}{2ix}((1+ix)^{2n+1} - (1-ix)^{2n+1}).$$

So it suffices to show that $(1 + i \tan \frac{k\pi}{2n+1})^{2n+1} = (1 - i \tan \frac{k\pi}{2n+1})^{2n+1}$. This holds because

$$\begin{aligned}
\frac{1 + i \tan \frac{k\pi}{2n+1}}{1 - i \tan \frac{k\pi}{2n+1}} &= \frac{1 - (\tan \frac{k\pi}{2n+1})^2 + 2i \tan \frac{k\pi}{2n+1}}{1 + (\tan \frac{k\pi}{2n+1})^2} \\
&= \cos^2 \frac{k\pi}{2n+1} - \sin^2 \frac{k\pi}{2n+1} + 2i \sin \frac{k\pi}{2n+1} \cos \frac{k\pi}{2n+1} \\
&= \cos \frac{2k\pi}{2n+1} + i \sin \frac{2k\pi}{2n+1},
\end{aligned}$$

which is one of the $(2n+1)$th complex roots of 1.

6. Fix positive integers n, k such that $k \leq n-1$. A permutation $a_1, \cdots, a_n$ of the numbers $1, 2, \cdots, n$ is called a ***k-shuffle*** if $1, 2, \cdots, k$ occur in the correct order and $k+1, k+2, \cdots, n$ occur in the correct order. For example, the 2-shuffles of $1, 2, 3, 4$ are those permutations where 1 precedes 2 and 3 precedes 4, namely (omitting the commas) 1234, 1324, 1342, 3124, 3142, and 3412. For any distinct complex numbers $x_1, \cdots, x_n$, show that

$$\sum_{\substack{a_1, \cdots, a_n \\ \text{a } k\text{-shuffle}}} \frac{1}{(x_{a_1} - x_{a_2})(x_{a_2} - x_{a_3}) \cdots (x_{a_{n-1}} - x_{a_n})} = 0.$$

Solution. Let $S_k(n)$ be the set of all k-shuffles of $1, \cdots, n$. Clearly any k-shuffle must end either with k or with n; let $S_k(n)'$ and $S_k(n)''$ be the sets of k-shuffles of these two kinds. It suffices to show that

$$\begin{aligned}
&\sum_{\substack{a_1, \cdots, a_n \\ \in S_k(n)'}} (x_{a_1} - x_{a_2})^{-1} \cdots (x_{a_{n-1}} - x_{a_n})^{-1} \\
&\quad = (x_1 - x_2)^{-1} \cdots (x_{k-1} - x_k)^{-1} (x_{k+1} - x_{k+2})^{-1} \cdots (x_{n-1} - x_n)^{-1} (x_n - x_k)^{-1}, \\
&\sum_{\substack{a_1, \cdots, a_n \\ \in S_k(n)''}} (x_{a_1} - x_{a_2})^{-1} \cdots (x_{a_{n-1}} - x_{a_n})^{-1} \\
&\quad = (x_1 - x_2)^{-1} \cdots (x_{k-1} - x_k)^{-1} (x_{k+1} - x_{k+2})^{-1} \cdots (x_{n-1} - x_n)^{-1} (x_k - x_n)^{-1},
\end{aligned}$$

since the sum of the right-hand sides is clearly zero. We prove these equations by induction on n (they are trivial when $n = 2$). The two equations are related simply by replacing k by $n - k$ and swapping $x_1, \cdots, x_k$ and $x_{k+1}, \cdots, x_n$, so it suffices to prove the second one. If $k = n-1$, then the only element of $S_k(n)''$ is the trivial permutation, and the claim is obvious. Otherwise, $a_1, \cdots, a_n$ is in $S_k(n)''$ if and only if $a_n = n$ and $a_1, \cdots, a_{n-1}$ is in $S_k(n-1)$. Hence by the induction hypothesis,

$$\begin{aligned}
&\sum_{\substack{a_1,\cdots,a_n\\ \in S_k(n)''}} (x_{a_1}-x_{a_2})^{-1}\cdots(x_{a_{n-1}}-x_{a_n})^{-1}\\
&=\sum_{\substack{a_1,\cdots,a_{n-1}\\ \in S_k(n-1)'}} (x_{a_1}-x_{a_2})^{-1}\cdots(x_{a_{n-2}}-x_{a_{n-1}})^{-1}(x_k-x_n)^{-1}\\
&\quad+\sum_{\substack{a_1,\cdots,a_{n-1}\\ \in S_k(n-1)''}} (x_{a_1}-x_{a_2})^{-1}\cdots(x_{a_{n-2}}-x_{a_{n-1}})^{-1}(x_{n-1}-x_n)^{-1}\\
&=(x_1-x_2)^{-1}\cdots(x_{k-1}-x_k)^{-1}(x_{k+1}-x_{k+2})^{-1}\cdots(x_{n-2}-x_{n-1})^{-1}(x_{n-1}-x_k)^{-1}(x_k-x_n)^{-1}\\
&\quad+(x_1-x_2)^{-1}\cdots(x_{k-1}-x_k)^{-1}(x_{k+1}-x_{k+2})^{-1}\cdots(x_{n-2}-x_{n-1})^{-1}(x_k-x_{n-1})^{-1}(x_{n-1}-x_n)^{-1}.
\end{aligned}$$

The desired expression now follows from the identity

$$(x_{n-1}-x_k)^{-1}(x_k-x_n)^{-1}+(x_k-x_{n-1})^{-1}(x_{n-1}-x_n)^{-1}=(x_{n-1}-x_n)^{-1}(x_k-x_n)^{-1}.$$

7. Suppose we have m white balls and n black balls, indistinguishable apart from their colour. We put them in a bag to hide the colour, and then draw out b of the $m+n$ balls, chosen at random. For any a, let $P(a;b,m,n)$ denote the probability that at least a of these b balls are white. On the assumption that a and b are nonnegative integers satisfying $0 \le b \le m+n$, $0 \le a \le m$, and $0 \le b-a \le n$, prove that

$$P(a+1;b,m,n) < P(a+1;b+1,m+1,n+1) < P(a;b,m,n).$$

Solution. This result is proved in the paper 'On the comparison of two observed frequencies' by M. Phipps and E. Seneta, Biometrical Journal 43 (2001), no. 1, pp. 23–43.

8. Let A be the set of rational numbers r such that $0 < r < 1$. It is well known that A is *countable*, i.e. the elements of A can be listed $r_1, r_2, r_3, \cdots$ so that every element appears exactly once on the list. Given such a listing, we define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{\substack{n\ge 1\\ r_n \le x}} 2^{-n}.$$

a) Show that there exists a listing of A for which the corresponding function f takes no rational values other than 0 and 1.

b) Show that there exists a listing of A for which f takes infinitely many rational values.

Solution.

a) Express all the elements of A as fractions in lowest terms, and then list them by order of their denominators, and by order of numerators within ones with the same denominator:

$$\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{1}{4},\frac{3}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{1}{6},\frac{5}{6},\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7},\ldots$$

Note that there are $\phi(q)$ numbers on this list with denominator q. Let f be the corresponding function. It is obvious that $f(x) = 0$ for $x \le 0$, and $f(x) = \sum_{n\ge 1} 2^{-n} = 1$ for

$x \geq 1$. So assume $0 < x < 1$, and suppose for a contradiction that $f(x)$ is rational. Then the infinite binary expansion of $f(x)$ has some initial segment and then a repeating block of length N (if $f(x) = \frac{p}{2^k l}$ where $\gcd(l, 2p) = 1$, N is the multiplicative order of 2 in $\mathbb{Z}/l\mathbb{Z}$). But by definition of $f(x)$, the nth bit after the 'binary point' is 1 if $r_n \leq x$ and 0 otherwise. Thus for our listing, the first $\phi(2) = 1$ bit determines whether $\frac{1}{2} \leq x$; the next $\phi(3) = 2$ bits determine whether $\frac{1}{3} \leq x$ and $\frac{2}{3} \leq x$, and so on. Clearly if q is prime, the corresponding $q - 1$ bits consist of $\lfloor qx \rfloor$ ones followed by $q - 1 - \lfloor qx \rfloor$ zeroes. When q is sufficiently large both these numbers exceed N, contradicting the supposed periodicity.

b) Let $a_1 > a_2 > a_3 > \cdots$ be any infinite decreasing sequence of irrational numbers in the interval $(0, 1)$ whose limit is 0. Then A is the disjoint union $A_1 \cup A_2 \cup A_3 \cup \cdots$, where $A_1 = A \cap (a_1, 1)$ and $A_j = A \cap (a_j, a_{j-1})$ for all $j \geq 2$. Also $\mathbb{Z}^+$ is the disjoint union $N_1 \cup N_2 \cup N_3 \cup \cdots$, where $N_j = \{n \in \mathbb{Z}^+ \mid n \equiv 2^{j-1} \bmod 2^j\}$. It is clear that each set A_j and N_j is countably infinite. Hence we can define bijections $q_j : N_j \to A_j$ and put them together to define a bijection $q : \mathbb{Z}^+ \to A$ (that is, a listing as in the problem). If f is the corresponding function, then for all integers $k \geq 1$,

$$\begin{aligned} f(a_k) &= \sum_{\substack{n\geq 1 \\ q(n)\in A_{k+1}\cup A_{k+2}\cup\cdots}} 2^{-n} \\ &= \sum_{\substack{n\geq 1 \\ n\in N_{k+1}\cup N_{k+2}\cup\cdots}} 2^{-n} \\ &= \sum_{n\in 2^k\mathbb{Z}^+} 2^{-n} \\ &= \frac{1}{2^{2^k} - 1}. \end{aligned}$$

So the numbers $f(a_k)$ constitute the required set of infinitely many rational values of f.

9. Fix a positive integer n and let $x_1, \cdots, x_n$ be indeterminates. For any permutation $a_1, \cdots, a_n$ of $1, \cdots, n$, define a polynomial in $x_1, \cdots, x_n$:

$$\Pi_{a_1,\cdots,a_n} = (x_{a_1} - x_{a_2})(x_{a_1} + x_{a_2} - x_{a_3})(x_{a_1} + x_{a_2} + x_{a_3} - x_{a_4}) \cdots (x_{a_1} + x_{a_2} + \cdots + x_{a_{n-1}} - x_{a_n}).$$

Prove that each of these polynomials is a linear combination, with integer coefficients, of the polynomials attached to permutations where $a_1 = 1$.

Solution. We prove this by induction on n, it being trivial when $n = 1$. First suppose that $1 = a_j$ for $1 \leq j \leq n - 1$. In this case we observe that

$$\Pi_{a_1,\cdots,a_n} = \Pi_{a_1,\cdots,a_{n-1}}(x_{a_1} + x_{a_2} + \cdots + x_{a_{n-1}} - x_{a_n}).$$

By the result for $n-1$ applied to the indeterminates $x_1, x_{a_1}, \cdots, \widehat{x_{a_j}}, \cdots, x_{a_{n-1}}$, the polynomial $\Pi_{a_1,\cdots,a_{n-1}}$ is an integral linear combination of polynomials $\Pi_{1,b_2,\cdots,b_{n-1}}$ where $b_2, \cdots, b_{n-1}$ is a permutation of $\{a_i \mid 1 \leq i \leq n - 1, i \neq j\}$. For such polynomials we have

$$\Pi_{1,b_2,\cdots,b_{n-1}}(x_{a_1} + x_{a_2} + \cdots + x_{a_{n-1}} - x_{a_n}) = \Pi_{1,b_2,\cdots,b_{n-1},a_n},$$

so this gives the required linear combination. So we need only handle the case where $1 = a_n$; by symmetry, it will suffice to show that $\Pi_{2,3,\cdots,n,1}$ is an integral linear combination of polynomials $\Pi_{1,b_2,\cdots,b_n}$. This is obvious if $n = 2$, so assume $n \geq 3$. Now

$$\Pi_{2,3,\cdots,n,1} = (x_2 - x_3)\,\Pi_{2,4,\cdots,n,1}|_{x_2\mapsto x_2+x_3},$$

where on the right-hand side we have a polynomial in the indeterminates $x_2, x_4, \cdots, x_n, x_1$, but with $x_2 + x_3$ substituted for x_2. By the result for $n-1$ again, $\Pi_{2,4,\cdots,n,1}$ is an integral linear combination of polynomials $\Pi_{1,c_2,\cdots,c_{n-1}}$ where $\{c_2, \cdots, c_{n-1}\} = \{2, 4, \cdots, n\}$, and $2 = c_k$ say. Now let X be the product of all the factors in such a polynomial except $x_1+x_{c_2}+\cdots+x_{c_{k-1}}-x_2$, and let Y denote $x_1 + x_{c_2} + \cdots + x_{c_{k-1}}$. We have

$$\begin{aligned}(x_2 - x_3)\,\Pi_{1,c_2,\cdots,c_{n-1}}|_{x_2\mapsto x_2+x_3} &= X|_{x_2\mapsto x_2+x_3}(x_2 - x_3)(Y - x_2 - x_3)\\ &= X|_{x_2\mapsto x_2+x_3}(Y - x_2)(Y + x_2 - x_3)\\ &\quad - X|_{x_2\mapsto x_2+x_3}(Y - x_3)(Y + x_3 - x_2)\\ &= \Pi_{1,c_2,\cdots,c_{k-1},2,3,c_{k+1},\cdots,c_{n-1}} - \Pi_{1,c_2,\cdots,c_{k-1},3,2,c_{k+1},\cdots,c_{n-1}},\end{aligned}$$

which is of the required form.

10. Fix an integer $n \geq 2$. Determine for which real numbers c the following polynomial has n real roots (counting multiplicities):

$$x^n + cx^{n-1} + \binom{c}{2}x^{n-2} + \binom{c}{3}x^{n-3} + \cdots + \binom{c}{n},$$

where $\binom{c}{s}$ means $\frac{c(c-1)(c-2)\cdots(c-s+1)}{s!}$.

Solution. We will show that the set of c satisfying this condition is as follows: if $n = 2$, it is the interval $[0, 2]$; if $n = 3$, it is the union $\{0\} \cup [1, 2] \cup \{3\}$; if $n \geq 4$, it is the finite set $\{0, 1, \cdots, n\}$.

In the $n = 2$ case, we just need to establish for which c the quadratic $x^2 + cx + \frac{c(c-1)}{2}$ has real roots. The discriminant is $c^2 - 2(c^2 - c) = c(2 - c)$, which is nonnegative precisely when $c \in [0, 2]$. From now on we assume $n \geq 3$. Denote the polynomial in question by f_c.

Firstly, note that if $c \in \{0, 1, \cdots, n\}$, then $f_c(x) = x^{n-c}(x+1)^c$, so these values of c definitely work. So assume that $c \notin \{0, 1, \cdots, n\}$. The key observation is the following equation of polynomials:

$$(nx - c + n)f_c(x) = x(x + 1)f_c'(x) - \frac{c(c-1)\cdots(c-n)}{n!}. \qquad (3)$$

One way to prove this is by comparing $x^{-n}f_c(x)$ with the power series expansion of $(1 + x^{-1})^c$ in the variable x^{-1}. More directly, we can simply find the coefficients of the powers of x on both sides of (3). Both sides have leading term nx^{n+1}. The constant term on the left-hand side is exactly the second term on the right-hand side, so constant terms also match. For $0 \leq s \leq n-1$, the coefficient of x^{n-s} on the left-hand side is

$$n\binom{c}{s+1} + (-c + n)\binom{c}{s} = \frac{1}{s+1}\binom{c}{s}[n(c - s) + (-c + n)(s + 1)],$$

while the coefficient of x^{n-s} on the right-hand side is

$$(n - s - 1)\binom{c}{s+1} + (n - s)\binom{c}{s} = \frac{1}{s+1}\binom{c}{s}[(n - s - 1)(c - s) + (n - s)(s + 1)].$$

These are clearly the same.

Since we have assumed the constant term on the right-hand side of (3) is nonzero, we see instantly that f_c and f_c' have no common root, i.e. f_c has no repeated root. By elementary calculus, if $x_1 > x_2 > \cdots > x_k$ are the real roots of f_c, we must have $f_c'(x_1) > 0$, $f_c'(x_2) < 0$, $f_c'(x_3) > 0$, and so on. But (3) obviously implies that 0 and -1 are not roots of f_c, and within each interval $(-\infty, -1)$, $(-1, 0)$, $(0, \infty)$, the sign of $f_c'(x)$ is the same for all roots. We conclude that f_c has at most 3 roots (counting multiplicities). So for $n \geq 4$, we do not get any values of c outside $\{0, 1, \cdots, n\}$.

If $n = 3$ and $c \notin \{0, 1, 2, 3\}$ is such that f_c does have 3 real roots, then by the above reasoning there must be one in each of the intervals $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$, and moreover $f_c'(x_1) > 0$ where x_1 is the root in $(0, \infty)$. Equation (3) then implies that $c(c-1)(c-2)(c-3) > 0$. Moreover, $f_c'(x) = 3x^2 + 2cx + \frac{c(c-1)}{2}$ must have two real roots, so its discriminant $2c(3 - c)$ is positive. We conclude that $1 < c < 2$. Conversely, assume $1 < c < 2$, and let $u < v$ be the roots of f_c'; we have

$$f_c'(\frac{c}{3} - 1) = 3(\frac{c}{3} - 1)^2 + 2c(\frac{c}{3} - 1) + \frac{c(c-1)}{2} = \frac{3}{2}(c-1)(c-2) < 0,$$

so $u < \frac{c}{3} - 1 < v$. Substituting $x = u$ in (3), we see that $(3u - c + 3)f_c(u) < 0$, which means that $f_c(u) > 0$; similarly, $f_c(v) < 0$. Hence f_c has three real roots.