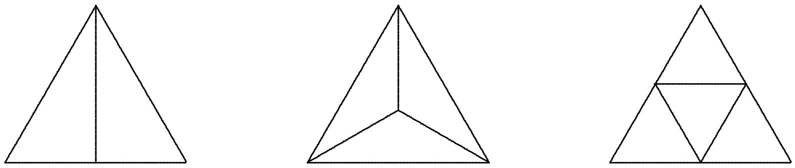
Sydney University Mathematical Society
Problems Competition 2004

Solutions.

1. We can partition an equilateral triangle into two, three or four congruent triangles, as the following diagram shows:



Show that it is not possible to partition it into five congruent triangles.

Solution. Let us call the original triangle T, and let ℓ be the length of each side of T. Suppose that we can partition T into 5 congruent triangles $T_1, \ldots, T_5$. Each of the three vertices of T must be a vertex of one or more of the T_i's. In addition, there may be vertices of the T_i's which are on the sides of T (but are not vertices of T), and there may be vertices of the T_i's which are in the interior of T. Let I denote the number of vertices of the T_i's which are in the interior of T, and let S be the number of vertices of the T_i's which are on the sides of T, but are not vertices of T.

If v is one of the I interior vertices, then the angles formed by T_i's at v add up to 360°. If v is one of the S side vertices, then the angles formed by T_i's at v add up to 180°. If v is one of the three vertices of T, then the angles formed by T_i's at v add up to 60°. The angles of the 5 triangles add up to $5 \times 180 = 900°$, and so we have the formula

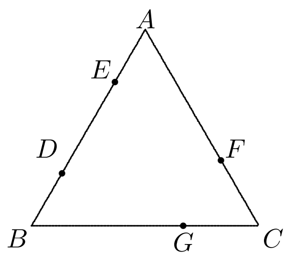
$$360 \times I + 180 \times E + 3 \times 60 = 900,$$

which simplifies to $2I + E = 4$. The non-negative integer solutions of this equation are $(I, E) = (2, 0)$, $(1, 2)$ and $(0, 4)$.

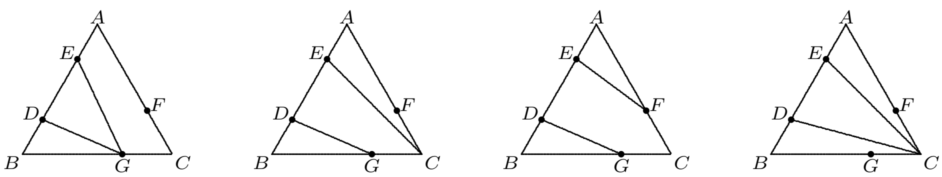
The possibility $E = 0$ is excluded. For this would mean that any side of T is a side of some T_i. But the triangles T_i are congruent, and so each one would have a side of length ℓ. But there are only three unordered pairs (P, Q) of points of T such that $\text{dist}(P, Q) = \ell$, namely the pairs of vertices of T.

For the same reason, the possibility $E = 2$ is excluded, because then at least one side of T has no vertex of any T_i in its interior, and so would be a side of some T_i.

So we must have $I = 0$ and $E = 4$, and for the same reasons as discussed in the previous two paragraphs, each side of T must have at least one vertex of a T_i in its interior. Therefore one side of T has two side vertices of T_i's, and the other two sides have one side vertex each. So we are in a situation like this:



The vertices $A, \ldots, G$ are the vertices of the 5 triangles. Interchanging the roles of D and E if necessary, there are four cases, according to which other vertices D and E are joined to:



We want to show that the first of these cases must hold. So suppose otherwise.

We first exclude the last of these cases. In that case, since there are no interior vertices, there is no edge of any T_i from A to G, and so the angle 60° at A must be one of the angles of one of the T_i's. Also, $\triangle DEC$ must be one of the five triangles T_i, and so one of its angles must be 60°. But if we write x°, y° and z° for the angles $\angle ACE$, $\angle ECD$ and $\angle DCB$, respectively, then

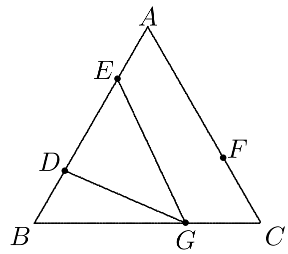
$$\begin{aligned}
&\angle DEC = 60^\circ + x^\circ > 60^\circ,\\
&\angle EDC = 60^\circ + z^\circ > 60^\circ, \text{ and}\\
&\angle ECD = y^\circ < 60^\circ,
\end{aligned}$$

so that no angle of $\triangle DEC$ is 60°, a contradiction.

The third of the above possibilities is excluded, because we suppose that we are not in the first case, and so there are no edges from D to F nor from E to G. So we must have edges from D to C and from E to C, so that we are also in the fourth case, which has been excluded.

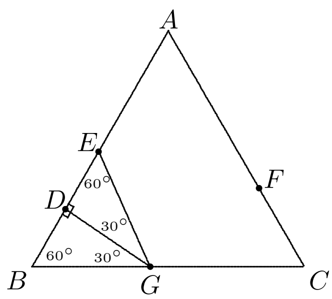
Similarly the second case is excluded, because we suppose that we are not in the first case, and so there is no edge from E to G, so there must be one from from D to C, so that once again we are also in the fourth case, which has been excluded.

So consider now the first of the above cases:



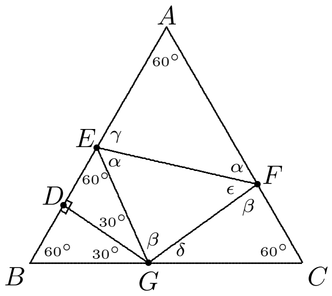
Then $\triangle BDG$ and $\triangle DEG$ must be two of the T_i's, and therefore congruent. So $\angle EDG$ must be one of the angles in $\triangle BDG$. But $\angle EDG = \angle DBG + \angle BGD$, so that the only angle of $\triangle BDG$ which $\angle EDG$ can equal is $\angle BDG$. Since $\angle EDG + \angle BDG = 180^\circ$, we must have $\angle EDG = \angle BDG = 90^\circ$. Hence $\angle BGD = 30^\circ$. So the values of $\angle DEG$ and $\angle DGE$ are 30° and 60°, or vice versa. If $\angle DEG = 30^\circ$, then in the congruence between $\triangle BDG$ and $\triangle GDE$, the edges BD and GD correspond, so that $\triangle BDG$ is isosceles, which is clearly not the case. So $\angle DEG$ must be 60°, and $\angle DGE$ must be 30°.

So we must be in the following situation:



There is no edge of any T_i joining A and G because otherwise $\triangle AEG$ would be one of the T_i's and so congruent to $\triangle BDG$. But then $\angle AEG = 120^\circ$ would be one of the angles of $\triangle BDG$, which is not the case. Similarly, there is no edge of any T_i joining C and E. So we must have edges from E to F and from G to F. Since the sides EG and AC are parallel, we know that $\angle FEG = \angle AFE, = \alpha$, say. For the same reason,

$\angle EGF = \angle CFG, = \beta$, say. Label other angles as in the diagram:



Then $\{\alpha, \gamma\} = \{30, 90\}$, $\{\beta, \delta\} = \{30, 90\}$, and $\alpha + \beta + \epsilon = 180$. The possibility $\alpha = \beta = 30$ is therefore impossible, because then the last equation would imply that $\epsilon = 120$. Also, the possibility $\alpha = \beta = 90$ is impossible, because then that equation would imply that $\epsilon = 0$. So $(\alpha, \beta) = (30, 90)$ or $(90, 30)$ (and in either case, $\epsilon = 60$).

The five triangles T_i all have an angle of $90°$. Let d denote the common length of their hypothenuses. Then the lengths of the sides of the T_i's next to an angle of $60°$ is $d/2$, and the lengths of the sides of the T_i's next to an angle of $30°$ is $d\sqrt{3}/2$. So if $(\alpha, \beta) = (30, 90)$, then $|GC| = d = |BG|$, and $|EA| = d/2 = |BD| = |DE|$. But then $|BC| = 2d$ and $|AB| = 3d/2$, contradicting the hypothesis that T is equilateral. If instead $(\alpha, \beta) = (90, 30)$, then $\delta = 90$, and so $|CF| = d = |AE|$ and $|AF| = d/2$, so that $|AB| = 2d$ and $|AC| = 3d/2$, again a contradiction. This completes the solution.

2. For which positive integers n is $2^{n-1} + 1$ divisible by n?

Solution. The only n which divides $2^{n-1} + 1$ is $n = 1$. More generally, let us show that if $a \geq 2$ is an even integer, then 1 is the only integer $n \geq 1$ such that n divides $a^{n-1} + 1$.

So suppose that $a \geq 2$ is even, and that $n > 1$ divides $a^{n-1} + 1$. Then n must be odd, since $a^{n-1} + 1$ is odd.

Let's now show that for any integer $a \geq 1$, the only odd integer n such that n divides $a^{n-1} + 1$ is $n = 1$. To see this, write $n - 1 = 2^s t$, where s, t are integers and t is odd. Then $s > 0$ because n is odd.

Now let p be a prime number which divides n. Since n is odd, we know that $p \neq 2$. Then p divides n and so p divides $a^{n-1} + 1$. Hence $a^{n-1} \equiv -1 \pmod{p}$, and therefore $a^{2(n-1)} \equiv 1 \pmod{p}$. In particular, p does not divide a. Let $h = h_p$ be the smallest positive integer such that $a^h \equiv 1 \pmod{p}$. It is well-known that for an integer $m \geq 1$, $a^m \equiv 1 \pmod{p}$ if and only if m is divisible by h. Another well-known fact is Fermat's Theorem, which says that $a^{p-1} \equiv 1 \pmod{p}$. It follows that

(i) h divides $p - 1$,

(ii) h divides $2(n - 1)$,

(iii) h does not divide $n - 1$.

So if we write $n - 1 = qh + r$, where q, r are integers and $0 \leq r < h$, then $r \neq 0$. However, h divides $2(n - 1) - 2qh = 2r$. Hence $2r$ is a multiple hk of h. Since $0 < r < h$, k must be 1, and so $(2q + 1)h = 2(n - 1) = 2^{s+1}t$. So h is divisible by 2^{s+1}, and so by (i), $p - 1$ is divisible by 2^{s+1}. Thus for each prime divisor p of n we can write $p = 1 + 2^{s+1}a_p$ for some integer a_p. Now n is a product $p_1^{m_1} \cdots p_r^{m_r}$, and since each p_j has the form $1 + 2^{s+1}a_j$, also $n = 1 + 2^{s+1}a$ for some integer a. But then $n - 1$ is divisible by 2^{s+1}, contrary to the defining property of s. This contradiction shows that n cannot exist.

If $a > 1$ is odd, then there are integers $n > 1$ (necessarily even, as we have just shown) such that n divides $a^{n-1} + 1$. For example, if n is an even divisor of $a + 1$, then n divides $a^{n-1} + 1$. One sees this easily by writing $a + 1 = nk$ and expanding $a^{n-1} + 1 = (nk - 1)^{n-1} + 1$ by the binomial theorem. This is not the full story, however. For example, $3^{n-1} + 1$ is divisible by n for $n = 28$ and $n = 532$.

3. Suppose that M is an $n \times n$ matrix of 0's and 1's with the property that in each row the 1's are adjacent to each other. For example,

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Show that $\det(M)$ is either 0, 1, or -1.

Solution. The proof is by induction on n. If $n = 1$, then $M = (0)$ or $M = (1)$, and so $\det(M)$ is either 0 or 1.

Now suppose that $n > 1$ and that the result has been proved for $(n-1) \times (n-1)$ matrices. Let M be an $n \times n$ matrix with the stated property. Consider the first column of M. If all the entries $m_{i,1}$ in that column are 0, then $\det(M) = 0$. So assume that at least one of the entries $m_{i,1}$ is 1. Amongst the i's for which $m_{i,1} = 1$, choose the i such that the number of 1's in row i is minimal. If this i is not 1, interchange rows 1 and i in M. This results in a new matrix, still denoted M, which still has the stated property, and whose determinant is -1 times the determinant of the matrix we started with. So now $m_{1,1} = 1$, and the number of 1's in row 1 is less than or equal to the number of 1's in any row i for which $m_{i,1} = 1$. Now for each row i such that $m_{i,1} = 1$, replace row i by row i minus row 1. This results in a new matrix, still denoted M, which has the same determinant as the M we had before, and which still has the stated property, because in each of the row operations, we subtracted a row of the form $(1, \ldots, 1, 0, \ldots, 0)$ (with its last 1 in column k, say) from a row of the form $(1, \ldots, 1, 0, \ldots, 0)$ (with its last 1 in column k', say, where $k' \geq k$), resulting in a row $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, whose 1's are all adjacent. After we have done these row operations, M has a 1 in position $(1, 1)$ and a 0 in all other positions of its first column. Hence $\det(M) = \det(M')$, where M' is the $(n-1) \times (n-1)$ matrix obtained from M by deleting row 1 and column 1. Since M' has the stated property, we know that $\det(M') \in \{0, 1, -1\}$ by the induction hypothesis. Hence $\det(M) = \det(M') \in \{0, 1, -1\}$ too.

4. Back in 1999, the following might have made a nice SUMS problem: "Consider the sum $1 - 1/2 + 1/3 - 1/4 + \cdots + 1/1331 - 1/1332$, written in reduced form as m/n. Show that m is divisible by 1999." Solve this problem, and formulate a similar problem for some years in the near future.

Solution. Consider the sum

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1} - \frac{1}{2k}.$$

We can write this

$$\begin{aligned}
&\Big(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2k-1} + \frac{1}{2k}\Big) - 2\Big(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\Big) \\
&= \Big(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2k-1} + \frac{1}{2k}\Big) - \Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}\Big) \\
&= \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{3} + \cdots + \frac{1}{2k} \\
&= \sum_{j=1}^{k/2} \Big(\frac{1}{k+j} + \frac{1}{2k+1-j}\Big) \quad \text{if } k \text{ is even} \\
&= \sum_{j=1}^{k/2} \frac{3k+1}{(k+j)(2k+1-j)}.
\end{aligned}$$

This last expression may be written $(3k+1)a/b$, where b is the product of all the terms $k+j$ and $2k+1-j$, for $j = 1, \ldots, k/2$. All of these terms are smaller than $3k+1$, so if $3k+1$ is prime, then when $(3k+1)a/b$ is put in reduced form, m/n, say, then the factor $3k+1$ is not cancelled from the numerator, and so m is divisible by $3k+1$.

Applying this with $k = 666$, the 1999 problem is solved. The next prime after 2004 is 2011, which equals $3k+1$ for $k = 670$, and so a similar problem can be posed in 2011, starting from the sum $1 - 1/2 + 1/3 - 1/4 + \cdots + 1/1339 - 1/1340$, and again in 2017. In $2027 = 3(675) + 2$, we can modify the problem a little more, starting from

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1}.$$

We can write this

$$\begin{aligned}
&\Big(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2k-1}+\frac{1}{2k}+\frac{1}{2k+1}\Big)-2\Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\Big)\\
&=\Big(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}\Big)-\Big(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\\
&=\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{3}+\cdots+\frac{1}{2k+1}\\
&=\sum_{j=1}^{(k+1)/2}\Big(\frac{1}{k+j}+\frac{1}{2k+2-j}\Big)\quad\text{if } k \text{ is odd}\\
&=\sum_{j=1}^{(k+1)/2}\frac{3k+2}{(k+j)(2k+2-j)}.
\end{aligned}$$

This will have reduced form m/n for m divisible by $3k+2$, provided that $3k+2$ is prime.

Since any prime number is either 2 or 3, or equals $3k+1$ (with k even) or $3k+2$ (with k odd) for some positive integer k, the above shows that a question like this could have been set in any prime year except the years 2 and 3.

5. Find a closed form expression for

$$\sum_{k=0}^{m}\Big(\frac{n!}{k!}\Big)^2\frac{1}{(n-2k)!4^k}$$

where $m=\lfloor\frac{n}{2}\rfloor$.

Solution. We give two solutions. Write the above as

$$\frac{n!}{2^n}\sum_{k=0}^{m}\frac{n!}{k!k!(n-2k)!}2^{n-2k}$$

By the Multinomial Theorem, this is the constant term in

$$\frac{n!}{2^n}\left(x^2+2+x^{-2}\right)^n.$$

But

$$\frac{n!}{2^n}\left(x^2+2+x^{-2}\right)^n=\frac{n!}{2^n}\left(x+x^{-1}\right)^{2n},$$

and the Binomial Theorem tells us that the constant term on the right is

$$\frac{n!}{2^n}\binom{2n}{n}=\frac{(2n)!}{2^n n!}\qquad\big(=(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1\big).$$

Here is a second solution, which gives a combinatorial interpretation of the identity. Suppose that you consider strings of 0's and 1's of length $2n$, and you want to know how many strings have exactly n 1's. On the one hand, the answer is clearly $\binom{2n}{n}$. On the other hand, you can divide the $2n$ 0's and 1's in a string into n consecutive pairs. Some of these pairs will be 00's, some 10's, some 01's and some 11's. Suppose there are k 00's. Then no matter how many 10's and 01's you have, they contribute the same number of 0's and 1's, and so in order to have n 1's (and therefore n 0's) you must also have k 11's. So $2k\le n$ must hold. To count the number of strings of length $2n$ containing exactly n 1's, we sum over the k's satisfying $0\le k\le n/2$ the number $N_{n,k}$ of such strings in which there are exactly k 00's. Having chosen the locations of these k pairs in $\binom{n}{k}$ ways, we can choose the locations of the k pairs 11 in $\binom{n-k}{k}$ ways. For each of the remaining $n-2k$ locations, you have 2 choices: 10 or 01. So there are 2^{n-2k} choices for "filling" these $n-2k$ places. So

$$N_{n,k}=\binom{n}{k}\binom{n-k}{k}2^{n-2k}=\frac{n!2^{n-2k}}{(k!)^2(n-2k)!}.$$

We sum $N_{n,k}$ over $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ to get the total number of strings in which there are exactly n 1's. Hence

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n! 2^{n-2k}}{(k!)^2 (n-2k)!} = \binom{2n}{n}.$$

Multiplying both sides by $n! 2^{-n}$, we get the formula to be proved.

6. Let $A \subset \mathbb{R}$. Call $x \in A$ an *interior point* of A if there is an $\epsilon > 0$ so that $(x-\epsilon, x+\epsilon) \subset A$. Let $\mathrm{Int}(A)$ denote the set of interior points of A. Let A^c denote the complement of A in $\mathbb{R}$. Starting from a set, we can (i) form its interior, or (ii) take its complement. How many different sets can be obtained from a given set A by successively applying the operations (i) and (ii)?

Solution. There are at most 14 different sets you can obtain from a given set A by successive operations of taking the interior and complement. It is easiest to explain this if we introduce some terminology. Let us call a set $A \subset \mathbb{R}$ *open* if $\mathrm{Int}\, A = A$. It is easy to see that the interior of any set is open, that is, $\mathrm{Int}(\mathrm{Int}\, A) = \mathrm{Int}\, A$. If $A \subset \mathbb{R}$ and if $O \subset A$ is open, then $O \subset \mathrm{Int}\, A$. Hence $\mathrm{Int}\, A$ is the largest open set contained in A. If $A \subset B$, then $\mathrm{Int}\, A \subset \mathrm{Int}\, B$.

We call $A \subset \mathbb{R}$ *closed* if its complement A^c is open. If $A \subset \mathbb{R}$, then $\mathrm{Int}\big(A^c\big) \subset A^c$, and so $(\mathrm{Int}(A^c))^c \supset A$. The set $(\mathrm{Int}(A^c))^c$ is closed because its complement $\mathrm{Int}(A^c)$ is open. If F is any closed set containing A, then $F^c \subset A^c$, and F^c is open. Hence $F^c \subset \mathrm{Int}(A^c)$, and so $F \supset (\mathrm{Int}(A^c))^c$. Hence the set $(\mathrm{Int}(A^c))^c$ is the smallest closed set containing A, and we call it the *closure* of A, and we denote it by $\overline{A}$. It is easy to see that $\overline{A} \subset \overline{B}$ if $A \subset B$. Of course $\overline{A}$ is always closed, and so $\overline{\overline{A}} = \overline{A}$ for any $A \subset \mathbb{R}$.

We need the following fact:

$$\overline{\mathrm{Int}\big(\overline{O}\big)} = \overline{O} \quad \text{if } O \subset \mathbb{R} \text{ is open.} \tag{1}$$

To see this, note first that $\mathrm{Int}\big(\overline{O}\big) \supset \mathrm{Int}(O) = O$, and so $\overline{\mathrm{Int}\big(\overline{O}\big)} \supset \overline{O}$. On the other hand, $\mathrm{Int}\big(\overline{O}\big) \subset \overline{O}$, and so $\overline{\mathrm{Int}\big(\overline{O}\big)} \subset \overline{\overline{O}} = \overline{O}$.

In terms of our given operations of taking complements and interiors, (1) says that

$$\mathrm{Int}((\mathrm{Int}((\mathrm{Int}((\mathrm{Int}\, A)^c))^c))^c) = \mathrm{Int}((\mathrm{Int}\, A)^c) \quad \text{for any } A \subset \mathbb{R}. \tag{2}$$

If we are given a set A, and we start by taking its interior, then to next get a new set, we must take this last set's complement. To then get a new set, we must take this last set's interior, and so on:

$$\begin{aligned} A \to \mathrm{Int}\, A \to (\mathrm{Int}\, A)^c &\to \mathrm{Int}((\mathrm{Int}\, A)^c) \\ &\to (\mathrm{Int}((\mathrm{Int}\, A)^c))^c \\ &\to \mathrm{Int}((\mathrm{Int}((\mathrm{Int}\, A)^c))^c) \\ &\to (\mathrm{Int}((\mathrm{Int}((\mathrm{Int}\, A)^c))^c))^c, \end{aligned} \tag{3}$$

but when we next take the interior, we get back to $\mathrm{Int}((\mathrm{Int}\, A)^c)$, by (2). We see seven sets in the above string (3) of arrows.

If from our given set A we instead start by taking its complement, then we get at most seven new sets, namely the sets

$$A^c,\ \mathrm{Int}(A^c),\ (\mathrm{Int}(A^c))^c, \ldots \tag{5}$$

obtained from the sets in (3) by replacing A by A^c. So all together, we get at most 14 sets from a given set A. To see that it is possible to in fact get 14 different sets, consider the following example:

$$A = \{x \in (0,1) : x \text{ is rational}\} \cup (2,3) \cup (3,4) \cup \{5\}.$$

Then

$$\begin{aligned} \mathrm{Int}\, A &= (2,3) \cup (3,4), \\ (\mathrm{Int}\, A)^c &= (-\infty, 2] \cup \{3\} \cup [4, \infty), \\ \mathrm{Int}((\mathrm{Int}\, A)^c) &= (-\infty, 2) \cup (4, \infty), \qquad * \\ (\mathrm{Int}((\mathrm{Int}\, A)^c))^c &= [2,4], \\ \mathrm{Int}((\mathrm{Int}((\mathrm{Int}\, A)^c))^c) &= (2,4), \\ (\mathrm{Int}((\mathrm{Int}((\mathrm{Int}\, A)^c))^c))^c &= (-\infty, 2] \cup [4, \infty), \end{aligned}$$

and taking interiors again we get back to the set marked by the asterisk. So we have seven distinct sets so far. The complement of A is

$$A^c = (-\infty, 0] \cup \{x \in (0,1) : x \text{ is irrational}\} \cup [1,2] \cup \{3\} \cup [4,5) \cup (5,\infty).$$

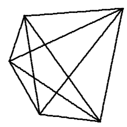
So

$$\begin{aligned}
\operatorname{Int} A^c &= (-\infty,0) \cup (1,2) \cup (4,5) \cup (5,\infty),\\
(\operatorname{Int} A^c)^c &= [0,1] \cup [2,4] \cup \{5\},\\
\operatorname{Int}((\operatorname{Int} A^c)^c) &= (0,1) \cup (2,4), \qquad \dagger\\
(\operatorname{Int}((\operatorname{Int} A^c)^c))^c &= (-\infty,0] \cup [1,2] \cup [4,\infty)\\
\operatorname{Int}((\operatorname{Int}((\operatorname{Int} A^c)^c))^c) &= (-\infty,0) \cup (1,2) \cup (4,\infty)\\
(\operatorname{Int}((\operatorname{Int}((\operatorname{Int} A^c)^c))^c))^c &= [0,1] \cup [2,4],
\end{aligned}$$

and taking interiors again we get back to the set marked by the dagger. So starting from A^c we have made seven sets, all different from the seven sets we got earlier. So it is possible to get 14 different sets.

If you know what a metric space is, or more generally, what a topological space is, then it is clear that the above result is true if $\mathbb{R}$ is replaced by any metric or topological space, and in this more general context it is due to Kuratowski.

7. Let f_n denote the number of non-overlapping regions into which the interior of a convex n-gon is divided by its diagonals. You should assume that no three diagonals meet at a point inside the n-gon. For example, the diagonals divide the following 5-gon into 11 non-overlapping regions.



Find a formula for f_n.

Solution. We show that

$$f_n = \frac{(n-1)(n-2)}{2} + \binom{n}{4} \quad \text{for } n = 3,4,\dots \tag{1}$$

by induction on n. Clearly $f_3 = 1$ and $f_4 = 4$, and so this formula is valid for $n = 3, 4$ (defining $\binom{3}{4} = 0$ as usual).

Now suppose that $n \geq 4$ and that we have proved the formula for any convex n-gon. Consider a convex $(n+1)$-gon X, and label its vertices $V_1, \dots, V_{n+1}$ in anti-clockwise order. Join the diagonal from V_1 to V_n. This divides our $(n+1)$-gon into a triangle, $\triangle V_1 V_n V_{n+1}$, and a convex n-gon Y with vertices $V_1, V_2, \dots, V_n$. The diagonals of Y divide it into f_n non-overlapping regions. The diagonals of X consist of those of Y together with the diagonals joining V_{n+1} to each of $V_2, \dots, V_{n-1}$. Suppose that $2 \leq k \leq n-1$, and consider the diagonal D_k joining V_{n+1} to V_k. There are $k-1$ vertices of X on one side of D_k, namely $V_1, \dots, V_{k-1}$, and $n-k$ vertices of X on the other side of D_k, namely $V_{k+1}, \dots, V_n$. The diagonal D_k therefore crosses $(k-1)(n-k)$ diagonals of Y, namely the diagonals from V_i to V_j for $i = 1, \dots, k-1$ and $j = k+1, \dots, n$. Imagine moving along D_k from V_{n+1} to V_k. Let $P_1, \dots, P_{(k-1)(n-k)}$ denote the successive crossing points along D_k. So D_k meets the diagonal $V_1 V_n$ of X at the point P_1, dividing a triangular region into two triangular regions, therefore adding a region. As we move from P_1 to P_2, we divide a region into 2, therefore adding a region. Continuing in this way, as we move from P_i to P_{i+1}, we add another region. Finally, as we go from $P_{(k-1)(n-k)}$ to V_k, we add another region. So all together, drawing the diagonal D_k adds $(k-1)(n-k)+1$ new regions.

We started with f_n regions in X, then added V_{n+1}, getting one more region (the triangle $\triangle V_1 V_n V_{n+1}$), and then for $k = 2, \dots, n-1$ we drew the diagonal D_k, thereby adding $(k-1)(n-k)+1$ new regions. So

$$f_{n+1} = f_n + 1 + \sum_{k=2}^{n-1} \big((k-1)(n-k)+1\big).$$

Using $\sum_{k=1}^n k = n(n+1)/2$ and $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$, it is routine to see that

$$1 + \sum_{k=2}^{n-1}\big((k-1)(n-k)+1\big) = n-1+\sum_{k=1}^{n}(k-1)(n-k) = \frac{n^3-3n^2+8n-6}{6}.$$

So

$$f_{n+1} = f_n + \frac{n^3-3n^2+8n-6}{6},$$

and it is now a routine matter to verify (1) by induction.

8. Show that for any integers $m, n \geq 1$, the expression

$$\frac{m(m-1)}{(n+1)(mn+1)}\binom{mn+n}{n}$$

is an integer.

Solution. The solution uses three simple rules about divisibility:

(i) If $a \mid c$, $b \mid c$ and $\gcd(a,b)=1$, then $ab \mid c$.

(ii) If $a \mid bc$ and if $\gcd(a,b)=1$, then $a \mid c$.

(iii) If $a \mid b_1 c$ and if $\gcd(a,b_1) = \gcd(a,b_2)$, then $a \mid b_2 c$.

We start by observing that

$$\frac{m(m-1)}{(n+1)(mn+1)}\binom{mn+n}{n} = \frac{m(m-1)}{n(n+1)}\binom{mn+n}{n-1}.$$

So we have to show that $n(n+1)$ divides $m(m-1)\binom{mn+n}{n-1}$. Since $\gcd(n, n+1)=1$, by Rule (i) it is sufficient to show that each of n and $n+1$ divides $m(m-1)\binom{mn+n}{n-1}$.

The identity

$$(mn+1)\binom{mn+n}{n-1} = n\binom{mn+n}{n}$$

shows that n divides $(mn+1)\binom{mn+n}{n-1}$. Since $\gcd(n, mn+1)=1$, Rule (ii) implies that n divides $\binom{mn+n}{n-1}$. Hence n divides $m(m-1)\binom{mn+n}{n-1}$.

The identity

$$m(mn+1)\binom{mn+n}{n-1} = (n+1)\binom{mn+n}{n+1}$$

shows that $n+1$ divides $m(mn+1)\binom{mn+n}{n-1}$. Since

$$\gcd(n+1, mn+1) = \gcd(n+1, (n+1)m-(m-1)) = \gcd(n+1, m-1),$$

Rule (iii) implies that $n+1$ divides $m(m-1)\binom{mn+n}{n-1}$.

9. A *partition* of n is a non-increasing sequence $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ of positive integers such that $\sum_{i=1}^k a_i = n$. Let $\mathcal{P}(n)$ be the set of partitions of n. For example, $\mathcal{P}(3) = \{(3),(2,1),(1,1,1)\}$. A *path sequence* is a doubly infinite sequence $(p_i)_{i\in\mathbb{Z}} = \ldots p_{-2}p_{-1}p_0|p_1p_2\ldots$ of 0's and 1's. We use the $|$ to mark the position of p_0. The *weight* of a path sequence $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ is

$$\mathrm{wt}(\mathbf{p}) = \sum_{i:p_i=0} \#\{\, j<i \mid p_j = 1\,\}.$$

For example, the following path sequences all have weight 3:

$$\ldots 00001|110111\ldots \qquad \ldots 00010|101111\ldots \qquad \ldots 00100|011111\ldots$$

Moreover, up to a shift these are all of the path sequences of weight 3.
(1) Show that there is a bijection between the partitions of n and the path sequences $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ of weight n for which

$$\sum_{i\le 0} p_i = \sum_{i>0}(1-p_i) < \infty.$$

The examples above give such a bijection when $n = 3$.
(2) Suppose that $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ is a path sequence satisfying the condition in (1) and that $\mathbf{p}$ corresponds to the partition $\mathbf{a}$ under your bijection. Fix integers $a < b$ with $p_a = 1$ and $p_b = 0$. Let $\mathbf{p}'$ be the path sequence obtained by swapping p_a and p_b in $\mathbf{p}$; that is,

$$p'_i = \begin{cases} p_b, & \text{if } i = a,\\ p_a, & \text{if } i = b,\\ p_i, & \text{otherwise.}\end{cases}$$

Show that $\mathbf{p}'$ also satisfies the condition in (1) and describe the partition which corresponds to $\mathbf{p}'$ under your bijection.

Solution. Let $\mathcal{S}$ denote the set of doubly infinite sequences $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ of 0's and 1's such that

$$\sum_{i\le 0} p_i < \infty \quad\text{and}\quad \sum_{i\ge 1}(1-p_i) < \infty.$$

That is, $\mathcal{S}$ is the set of doubly infinite sequences $\mathbf{p} = (p_i)_{i\in\mathbb{Z}}$ of 0's and 1's for which there exist integers m, n such that $p_i = 0$ for all $i \le -m$ and $p_i = 1$ for all $i \ge n$.

If $\mathbf{p} \in \mathcal{S}$, let $\alpha(\mathbf{p}) = \sum_{i\le 0} p_i$, let $\beta(\mathbf{p}) = \sum_{i\ge 1}(1-p_i)$, and let $\gamma(\mathbf{p}) = \alpha(\mathbf{p}) - \beta(\mathbf{p})$. Suppose that we translate the sequence $\mathbf{p} = (p_i)$ one to the right, obtaining a new sequence $\mathbf{q} = (q_i)$. Then $q_i = p_{i-1}$ for all i. Hence

$$\alpha(\mathbf{q}) = \sum_{i\le 0} q_i = \sum_{i\le 0} p_{i-1} = \sum_{i\le -1} p_i = \alpha(\mathbf{p}) - p_0,$$

and

$$\beta(\mathbf{q}) = \sum_{i\ge 1}(1-q_i) = \sum_{i\ge 1}(1-p_{i-1}) = \sum_{i\ge 0}(1-p_i) = \beta(\mathbf{p}) + 1 - p_0,$$

so that $\gamma(\mathbf{q}) = (\alpha(\mathbf{p}) - p_0) - (\beta(\mathbf{p}) + 1 - p_0) = \gamma(\mathbf{p}) - 1$.

It follows that if $r \in \mathbb{Z}$, and if we translate $\mathbf{p} \in \mathcal{S}$ by r to the right (which means by $|r|$ to the left when r is negative), then the resultant sequence $\tau_r(\mathbf{p})$ satisfies $\gamma(\tau_r(\mathbf{p})) = \gamma(\mathbf{p}) - r$.

We are interested in the set $\mathcal{S}^*$, say, of $\mathbf{p} \in \mathcal{S}$ for which $\gamma(\mathbf{p}) = 0$. The above shows that for any $\mathbf{p} \in \mathcal{S}$, $\tau_r(\mathbf{p}) \in \mathcal{S}^*$ for $r = \gamma(\mathbf{p})$. Moreover, if $\mathbf{p}$ and $\mathbf{q}$ are in $\mathcal{S}^*$ and $\mathbf{q} = \tau_r(\mathbf{p})$ for some $r \in \mathbb{Z}$, then $r = 0$. In other words, each $\mathbf{p} \in \mathcal{S}$ is the translate of exactly one $\mathbf{q} \in \mathcal{S}^*$.

On the other hand, if we translate the sequence $\mathbf{p} = (p_i)$ one to the right, then it is easy to see that $\mathbf{q} = \tau_1(\mathbf{p})$ has the same weight as $\mathbf{p}$. Hence $\tau_r(\mathbf{p})$ and $\mathbf{p}$ have the same weight for each $r \in \mathbb{Z}$.

The bijection between partitions of n and path sequences of weight n in $\mathcal{S}^*$ is constructed as follows: Given a partition $a_1 \ge a_2 \ge \cdots a_k \ge 1$ of n, form the following finite string of 0's and 1's:

$$\overbrace{1\cdots 1}^{a_k\ 1'\text{s}}\ 0\ \overbrace{1\cdots 1}^{a_{k-1}-a_k\ 1'\text{s}}\ 0\ \cdots 0\ \overbrace{1\cdots 1}^{a_1-a_2\ 1'\text{s}}\ 0$$

Form a doubly infinite string of 0's and 1's from this by preceding it by infinitely many 0's and following it by infinitely many 1's, and by indexing the resultant sequence so that the first 1 is in position $i = 0$, say. Finally, let $\mathbf{p}$ be the unique element of $\mathcal{S}^*$ obtained from this string by translating it by the appropriate integer. The weight of $\mathbf{p}$ is the weight of the untranslated string, namely

$$a_k + a_{k-1} + \cdots + a_1 = n.$$

Conversely, if we are given a string $\mathbf{p}$ in $\mathcal{S}^*$ of weight n, obtain a finite string of 0's and 1's by discarding the infinitely many 0's before the first 1, and the infinitely many 1's following the last 0. Since $\mathbf{p}$ has weight $n \geq 1$, there is at least one 1 and at least one 0 left. Suppose that there are k 0's left. Let m_1 denote the number of 1's up to the first 0, let m_2 denote the number of 1's between the first and the second 0, and so on: for $j = 2, \ldots, k$, m_j is the number of 1's between the $(j-1)$-st and the j-th 0:

$$\overbrace{1\cdots1}^{m_1\ 1'\mathrm{s}}\ 0\overbrace{1\cdots1}^{m_2\ 1'\mathrm{s}}0\ \cdots0\overbrace{1\cdots1}^{m_k\ 1'\mathrm{s}}0.$$

Now $m_1 + \cdots + m_j$ is the number of 1's before the j-th 0, and so the weight n of $\mathbf{p}$ is

$$m_1 + (m_1 + m_2) + \cdots + (m_1 + m_2 + \cdots + m_k) = km_1 + (k-1)m_2 + \cdots + m_k.$$

We form the numbers $a_1, \ldots, a_k$ by setting

$$\begin{aligned} a_1 &= m_1 + m_2 + \cdots + m_{k-1} + m_k \\ a_2 &= m_1 + m_2 + \cdots + m_{k-1} \\ \vdots\ &= \ \vdots \\ a_{k-1} &= m_1 + m_2 \\ a_k &= m_1 \end{aligned}$$

Then $a_1 \geq a_2 \geq \cdots \geq a_k > 0$, and $a_1 + \cdots + a_k = n$. So we get a partition of n from $\mathbf{p}$.

It is evident that the procedures described above are mutually inverse, and give a bijection between the partitions of n and the elements of $\mathcal{S}^*$ having weight n.

Now suppose that $\mathbf{p} \in \mathcal{S}^*$ has weight n, and assume that $a < b$, $p_a = 1$ and $p_b = 0$. Let $\mathbf{p}'$ be as in the question. If $a, b \leq 0$, then $\alpha(\mathbf{p}') = \alpha(\mathbf{p})$ because the number of i's such that $i \leq 0$ and $p'_i = 1$ is the same as the number of i's such that $i \leq 0$ and $p_i = 1$. Similarly $\beta(\mathbf{p}') = \beta(\mathbf{p})$. Therefore

$$\gamma(\mathbf{p}') = \alpha(\mathbf{p}') - \beta(\mathbf{p}') = \alpha(\mathbf{p}) - \beta(\mathbf{p}) = \gamma(\mathbf{p}) = 0, \tag{1}$$

and so $\mathbf{p}'$ is in $\mathcal{S}^*$. The same is true if $a, b \geq 1$. So suppose that $a \leq 0$ and $b \geq 1$. Then $\alpha(\mathbf{p}') = \alpha(\mathbf{p}) - 1$ and $\beta(\mathbf{p}') = \beta(\mathbf{p}) - 1$, so again (1) holds. Therefore $\mathbf{p}' \in \mathcal{S}^*$.

Let $a'_1 \geq \cdots \geq a'_{k'} \geq 1$ be the partition which corresponds to $\mathbf{p}'$. Suppose that p_b is the t-th 0 to the right of the initial infinite string of 0's in $\mathbf{p}$. Suppose that p_a is between the $(s-1)$-st and the s-th 0, counting in the same way. Then $s \leq t$. From the block of m_s 1's between the original $(s-1)$-st and the s-th 0's, we have turned a 1 into a 0, leaving u 1's to the left of this 0 and v to the right, so that $u + v = m_s - 1$. In the new string $\mathbf{p}'$, the first $s-1$ 0's to the right of the initial infinite string of 0's are the same as in $\mathbf{p}$. Then the s-th 0 of $\mathbf{p}'$ is u to the right of the $(s-1)$-st 0, and the i-th 0 of $\mathbf{p}'$ is the same as the $(i-1)$-st 0 of $\mathbf{p}$ for $i = s+1, \ldots, t$. Then the t-th 0 of $\mathbf{p}$ has been turned into a 1, so the $(t+1)$-st 0 of $\mathbf{p}'$ is the same as the $(t+1)$-st 0 of $\mathbf{p}$, and indeed the i-th 0 of $\mathbf{p}'$ is the same as the i-th 0 of $\mathbf{p}$ for $i = t+1, \ldots, k$. The numbers m'_i of 1's between successive 0's in $\mathbf{p}'$ satisfy $m'_i = m_i$ for $i = 1, \ldots, s-1$, $m'_s = u$, $m'_{s+1} = v$, $m'_i = m_{i-1}$ for $i = s+2, \ldots, t$, $m'_{t+1} = m_t + m_{t+1} + 1$, and finally, $m'_i = m_i$ for $i = t+2, \ldots, k$.

One calculates that the partition $a'_1 \geq \cdots \geq a'_{k'} \geq 1$ corresponding to $\mathbf{p}'$ satisfies $k' = k$, and is given by

$$a'_i = a_{i+1} - 1 \quad \text{for } i = k-t+1, \ldots, k-s, \text{ and that } a'_{k-s+1} = a_{k-s+2} + u,$$

and that $a'_i = a_i$ for all other i's.

10. Let n be a positive integer. Let $a_1 \geq a_2 \geq \cdots \geq a_m$ be a partition of n (see the last question). Represent this partition as a left-justified array of boxes, with a_1 boxes in the first row, a_2 in the second, and so on, and label the boxes with 1 and -1 in a chess-board pattern, starting with a 1 in the top-left corner. Let c

be the sum of these labels. For instance, if $n = 11$ and the partition is $4, 3, 3, 1$, then $c = -1$, as one sees by summing the labels in the diagram:

1	−1	1	−1
−1	1	−1	
1	−1	1	
−1			

Prove that $n \geq c(2c - 1)$, and determine when equality occurs.

Solution. Suppose that $c \in \mathbb{Z}$. Then there is an integer n and a partition of n whose label sum is c. For if $c = 0$, we can take $n = 2$, whose partition $1 \geq 1$ has label sum 0. If $c > 0$, let $n = c(2c - 1)$, which has the partition $2c - 1 > 2c - 2 > \cdots > 2 > 1$ having label sum c. If $c < 0$, again let $n = c(2c - 1) = |c|(2|c| + 1)$, which has the partition $2|c| > 2|c| - 1 > \cdots > 2 > 1$ having label sum $-|c| = c$.

So for given c, the set S_c of integers $n \geq 1$ admitting a partition with label sum c is non-empty, and so has a minimal element. Choose $n \in S_c$ minimal, and choose a partition $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ of n with label sum c.

If $a_i = a_{i+1}$ for some $i < m$, then the labels in the i-th and $(i+1)$-st rows add to 0, and so deleting these two rows, we get a partition of $n - a_i - a_{i+1}$ which still has label sum c. This contradicts the minimality of n in S_c. Hence $a_i > a_{i+1}$ for each $i < m$.

If $a_i \geq a_{i+1} + 2$ for some $i < m$, then deleting the last two boxes from the i-th row corresponds to taking the partition $a_1 \geq \cdots \geq a_{i-1} \geq a_i - 2 \geq a_{i+1} \geq \cdots \geq a_m$ of $n - 2$. The two boxes deleting have one label of each sign, and so the label sum of the above partition of $n - 2$ is still c. This again contradicts the minimality of n in S_c. Hence $a_i \leq a_{i+1} + 1$ for each $i < m$. Therefore $a_i = a_{i+1} + 1$ for $i = 1, \ldots, m - 1$.

If $a_m \geq 2$, then deleting the last two boxes from the m-th row corresponds to taking the partition $a_1 \geq \cdots \geq a_{m-1} \geq a_m - 2$ of $n - 2$. Again the label sum remains equal to c, and the minimality of n is contradicted. Hence $a_m = 1$.

So the partition $a_1 \geq a_2 \geq \cdots \geq a_m$ must be the partition $m > m - 1 > \cdots > 2 > 1$, and so n must be $m(m + 1)/2$. The label sum is then $-m/2$ if m is even, and $(m + 1)/2$ if m is odd. Hence $m = -2c$ if $c < 0$ and $m = 2c - 1$ if $c > 0$. In both cases $n = m(m + 1)/2$ equals $c(2c - 1)$. By minimality of n in S_c, we see that $n' \geq n = c(2c - 1)$ for any integer n' in S_c. This is what we wanted to prove. We have seen above that there is only one partition of $n = c(2c - 1)$ which has label sum c.

The case $c = 0$ is a little special. Clearly $n = 2$ is the smallest positive integer admitting a partition with label sum 0, and in fact both partitions of 2 have label sum 0. It would perhaps be natural to allow $n = 0$, and the empty partition of 0, and decree that the label sum of this partition is 0. Then it is also true for $c = 0$ that there is a unique smallest integer $n \geq 0$ admitting a partition of n with label sum c, and the partition is unique.