SUMS PROBLEM COMPETITION, 2000

SOLUTIONS

1. The result is well known, and called Morley's Theorem. Many proofs are known. See for example H.S.M. Coxeter, "Introduction to Geometry", page 23.

2. If the number of vertices, edges and faces of a polyhedron are V, E and F, respectively, then Euler's formula states that $V - E + F = 2$. Let d_i be the number of edges coming out of the i-th vertex, and let e_j be the number of edges on the j-th face. Then $\sum_{i=1}^{V} d_i = 2E$, because each edge is counted twice, once for each of its endpoints. Also, $\sum_{j=1}^{F} e_j = 2E$ because each edge is common to exactly 2 faces. To prevent degeneracies, we must have $d_i, e_j \geq 3$ for each i, j. Hence

$$2E = \sum_{i=1}^{V} d_i \geq 3V \quad \text{and} \quad 2E = \sum_{j=1}^{F} e_j \geq 3F. \tag{1}$$

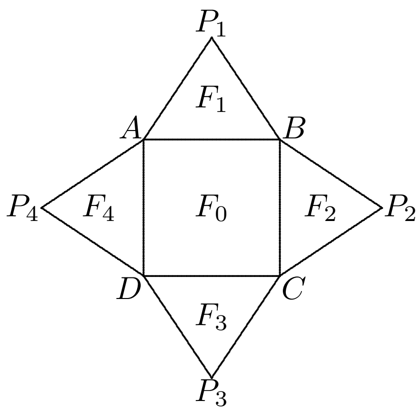
We now start answering the particular problems given:

a) If $E = 7$, then by (1), $V, F \leq \lfloor 2E/3 \rfloor = \lfloor 14/3 \rfloor = 4$. But Euler's formula tells us that $V + F = E + 2 = 9$. This is a contradiction, and so $E = 7$ is impossible.

b) Notice that $V \geq 4$, as otherwise we would have a planar figure. Hence $2E = \sum_{i=1}^{V} d_i \geq 3V \geq 12$, and so $E \geq 6$. If $k \geq 3$ is any integer, consider a pyramid with a k-gon as its base. This polyhedron has $E = 2k$ (and $V = F = k + 1$). So any even $E \geq 6$ is possible. If we use one of the sloping triangular faces of this pyramid as the base of a tetrahedron, we add 3 new edges, and so $E = 2k + 3$ (and $V = k + 2$, $F = k + 3$). So each odd $E \geq 9$ is possible. Hence the possible values of E are just the integers ≥ 6, except 7.

c) Suppose that $E = 11$. We shall show that up to isomorphism, there are precisely 4 possible polyhedra. Now $V + F = E + 2 = 13$ by Euler's formula. Also $V, F \leq \lfloor 2E/3 \rfloor = \lfloor 22/3 \rfloor = 7$ by (1). Hence the only possibilities are $(V, F) = (6, 7)$ and $(V, F) = (7, 6)$.

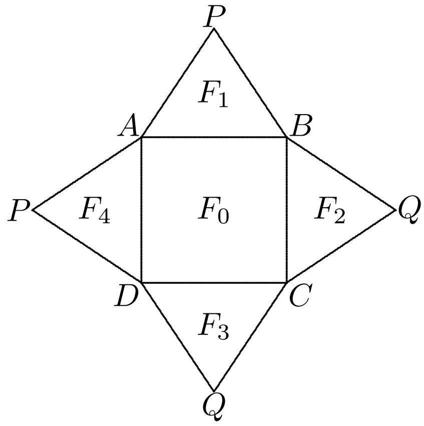
Case (i): $(V, F) = (6, 7)$. Now $\sum_{j=1}^{F} e_j = 2E = 22$ shows that one e_j equals 4, and the others equal 3. Let F_0 be the face with 4 sides. Call its vertices A, B, C and D. Let $F_1 \ldots, F_4$ be the second faces having edges AB, BC, CD and DA, respectively. These must all be triangles. Let $P_1, \ldots, P_4$ be their third vertices, as in the next diagram. We know that $P_1, \ldots, P_4$ cannot be A, B, C or D. Since $V = 6$, we have only 2 more vertices at our disposal, say P and Q. So the four vertices P_i cannot all be different; in fact, they must all be either P or Q.



Interchanging the labels P and Q if necessary, we may assume one of 3 possibilities:

1. $P_i = P$ for 2 i's and $P_i = Q$ for 2 i's,
2. $P_i = P$ for 3 i's and $P_i = Q$ for 1 i, or
3. $P_i = P$ for all 4 i's.

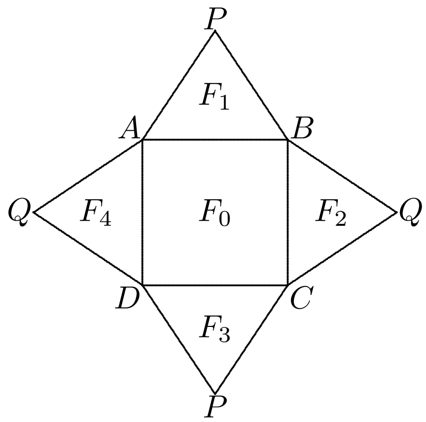
The first possibility has 2 subcases: 1(a). The two P's are "adjacent", and 1(b). The two P's are "opposite". In Case 1(a), we may assume that the two adjacent P's are P_1 and P_4:



Case 1(a).

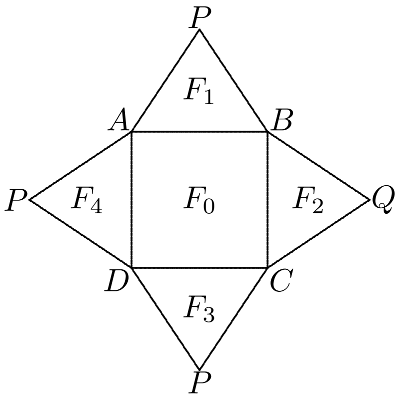
Now our polyhedron only has 11 edges, and 10 of them are already indicated in the last diagram: AB, BC, CD, DA, AP, BP, DP, BQ, CQ and DQ. We next show that the 11-th edge must be PQ. Suppose that it is not PQ. It cannot involve just A, B, C and D, so it must involve one of these vertices and one of P and Q. The possibilities are AQ or CP. Only one of these can be an edge, or else $E \geq 12$ would hold. Suppose that AQ is an edge. Consider the face $F_4 = ADP$. The edge DP of this face must lie on exactly one other face F, and F must be a triangle. Running through the possible third vertices of F besides D and P, we see that none is possible. Similarly, if CP is an edge, then the edge BQ of F_2 can only lie on one face, again a contradiction. So the 11-th edge of the polyhedron is PQ. Now all the edges of the figure are fixed. There is a polyhedron of this type. We can imagine it as inscribed in a cube, the face F_0 being the base, and the edge PQ joining two diagonally opposite vertices on the top of the cube.

Case 1(b) is impossible, because the following figure already shows 12 edges.



Case 1(b).

Consider Case 2:

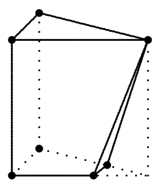


Case 1(a).

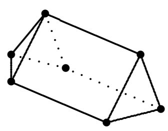
Again there are 10 edges shown. If the 11-th is not PQ, then it must be AQ or DQ. By symmetry, we may suppose that AQ is an edge, and so DQ is not an edge. Again by considering the second face having edge CQ, we get a contradiction. So PQ is the 11-th edge. All the edges are fixed now. The polyhedron must be the one defined in the solution of part (b) of this question, with a pyramid on the square base $ABCD$ and top at P, but then with a tetrahedron placed on the face BCP, its fourth vertex being Q.

Finally, let us eliminate Case 3. In that case, with all the P_i's equal to P, we have in the first of the above figures already a pyramid on a square base, with 8 edges. To get 11 edges, the remaining vertex Q must be connected to 3 of the vertices A, B, C, D and P. But then Q is connected to 2 adjacent vertices from amongst these 5, and the existing edge between these two vertices would have to belong to 3 faces, which is impossible.

Case (i): $(V, F) = (7.6)$. We can get the polyhedra of this type from the two just found by duality: we put a new vertex in the middle of each face, and join two new vertices if the corresponding faces have a common edge. This is what we get be dualizing the polyhedron of Case 1(a) above. It looks like a wedge from which we have made an oblique slice.



By dualizing the polyhedron of Case 2, we get the following polyhedron:



3. This problem is discussed in the book “Unsolved problems in number theory”, by R.K. Guy. The largest n for which SUMS, or indeed anyone, knows an example is $n = 5$. Here is such an example. Set

$$x_1 = 7\,442,\ x_2 = 28\,658,\ x_3 = 148\,583,\ x_4 = 177\,458,\ x_5 = 763\,442.$$

Then

$$\begin{aligned} x_1 + x_2 &= 190^2, & x_2 + x_4 &= 454^2,\\ x_1 + x_3 &= 395^2, & x_2 + x_5 &= 890^2,\\ x_1 + x_4 &= 430^2, & x_3 + x_4 &= 571^2,\\ x_1 + x_5 &= 878^2, & x_3 + x_5 &= 955^2,\\ x_2 + x_3 &= 421^2, & x_4 + x_5 &= 970^2. \end{aligned}$$

4. We show that the numbers n with the stated property are precisely the numbers of the form pq, where p and q are (possibly equal) prime numbers.

As a preliminary step, we first show that if $r \geq 2$ and if $n_1, \ldots, n_r \geq 2$ are integers, then $n_1 \times n_2 \times \cdots \times n_r \geq n_1 + \cdots + n_r$, with equality only if $r = 2$ and $n_1 = n_2 = 2$. To see this, first suppose that $r = 2$. Then

$$n_1 n_2 - (n_1 + n_2) = (n_1 - 1)(n_2 - 1) - 1 \geq 0,$$

with equality only when $n_1 = n_2 = 2$. Now suppose that $r \geq 3$, and that we have proved that $n_1 \times n_2 \times \cdots \times n_{r-1} \geq n_1 + \cdots + n_{r-1}$ if each n_i is at least 2. Given $n_1, \ldots, n_r \geq 2$, suppose that n_r is the least of the n_i's. Then

$$\begin{aligned} n_1 \times \cdots \times n_r &= (n_1 \times \cdots \times n_{r-1}) \times n_r\\ &\geq (n_1 + \cdots + n_{r-1}) \times n_r \quad \text{by the induction hypothesis,}\\ &\geq 2(n_1 + \cdots + n_{r-1})\\ &> n_1 + \cdots + n_{r-2} + 2n_{r-1} \quad \text{since } r \geq 3,\\ &\geq n_1 + \cdots + n_r. \end{aligned}$$

Now we return to the problem at hand. Suppose that n can be factored $n_1 n_2 n_3$, where $n_1, n_2, n_3 \geq 2$. Then by the statement just proved, $u := n_1 n_2 n_3 - (n_1 + n_2 + n_3)$ is strictly positive, and using u 1's in the next two equations, we can write

$$\begin{aligned} n &= n_1 \times n_2 \times n_3 \times 1 \times \cdots \times 1\\ n &= n_1 + n_2 + n_3 + 1 + \cdots + 1. \end{aligned}$$

But we can also write $n = (n_1 n_2) n_3$, and $n_1 n_2 \geq 4$ and $n_3 \geq 2$. So by the above statement, $v := n_1 n_2 n_3 - (n_1 n_2 + n_3)$ is strictly positive, and using v 1's in the next two equations, we can write

$$\begin{aligned} n &= (n_1 n_2) \times n_3 \times 1 \times \cdots \times 1\\ n &= (n_1 n_2) + n_3 + 1 + \cdots + 1. \end{aligned}$$

Thus whenever n may be written $n = n_1 n_2 n_3$, with the n_i's all at least 2, we can write n in at least two ways in the required way, and so n does not satisfy the given conditions. Hence any n which is divisible by 3 or more primes cannot satisfy the given conditions.

If $n = pq$ is the product of just 2 primes, then it is easy to see that, using $w = pq - (p+q)$ 1's in the next two equations,

$$\begin{aligned} n &= p \times q \times 1 \times \cdots \times 1\\ n &= p + q + 1 + \cdots + 1 \end{aligned}$$

shows that n satisfies the conditions (no 1's are used if $p = q = 2$).

To complete the proof, we observe that if $n = 1$ or if $n = p$, a prime, then n cannot satisfy the required conditions. For $1 = n_1 \times \cdots \times n_r$ can only happen if $n_1 = \cdots = n_r = 1$, in which case $1 < n_1 + \cdots + n_r = r$ since $r \geq 2$. Also $n = p = n_1 \times \cdots \times n_r$ can only happen if n_1, say, equals p, and the other n_i's equal 1. But then $n_1 + \cdots + n_r = p + (r-1) > p = n$. So a decomposition of $n = p$ of the required type is not possible.

5. If we multiply both sides of the equation $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$ by $x - y$, we get the equation $x^5 - y^5 = x - y$. That is, $x^5 - x = y^5 - y$. So to have four distinct numbers $x_1, \ldots, x_4$ with the given property is the same as having four distinct numbers $x_1, \ldots, x_4$ for which $x_1^5 - x_1 = \cdots = x_4^5 - x_4$. Let k be the common value of the $x_i^5 - x_i$. Then the equation $x^5 - x - k = 0$ must be satisfied by each of $x_1, \ldots, x_4$. We now show that, for any $k \in \mathbb{R}$, the equation $x^5 - x - k = 0$ has at most 3 solutions.

Let $f(x) = x^5 - x - k$. Then by Rolle's Theorem, between any two solutions of $f(x) = 0$ we must have a solution of $f'(x) = 0$. That is, $5x^4 - 1 = 0$. But the only real solutions of $5x^4 - 1 = 0$ are $\pm 1/5^{1/4}$. As $f'(x) = 0$ has only 2 solutions, Rolle's Theorem says that $f(x) = 0$ has at most 3 solutions.

6. Here are two different methods:

Method (i): Let $P(n)$ denote the set of partitions of n, i.e., the set of ordered r-tuples $(a_1, \ldots, a_r)$ of positive integers, where $r \geq 1$, $a_1 \geq \cdots \geq a_r$ and $a_1 + \cdots + a_r = n$. Thus $p_n = |P(n)|$. It is convenient to define $P(0)$ to consist of the empty tuple $()$, and set $p_0 = 1$. For all $n \geq 1$, consider the map $f : P(n) \to \bigcup_{i=1}^{n} P(n-i)$ defined by $f : (a_1, \ldots, a_r) \mapsto (a_2, \ldots, a_r) \in P(n - a_1)$, where in the case $r = 1$, the 1-tuple $(n) \in P(n)$ is mapped to the empty tuple $() \in P(0)$. This map is injective. For if f maps $(a_1, \ldots, a_r) \in P(n)$ and $(b_1, \ldots, b_s) \in P(n)$ to the same tuple, then $(a_2, \ldots, a_r) = (b_2, \ldots, b_s)$, so that $r = s$ and $a_i = b_i$ for $i = 2, \ldots, r$. Also $a_1 = b_1$ because $\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} b_i = n$.

The injectivity of f implies that $|P(n)| \leq \left|\bigcup_{i=1}^{n} P(n-i)\right|$. That is,

$$p_n \leq p_0 + \cdots + p_{n-1}. \tag{1}$$

We next show that $p_n \leq 2^{n-1}$ for all $n \geq 1$ by induction on n. Since $p_1 = 1 = 2^0$, this inequality is valid if $n = 1$. Assuming that $n \geq 2$ and the inequalities $p_k \leq 2^{k-1}$ are valid for $1 \leq k \leq n-1$, by (1) we get

$$p_n \leq 1 + 1 + \cdots + 2^{n-2} = 2^{n-1}.$$

Hence

$$p_1 + \cdots + p_n \leq 1 + 2^1 + \cdots + 2^{n-1} = 2^n - 1 < 2^n.$$

Method (ii). Let $C(n)$ denote the set of *compositions* of n, i.e., the set of ordered r-tuples $(a_1, \ldots, a_r)$ of positive integers, where $r \geq 1$ and $a_1 + \cdots + a_r = n$. Thus the condition $a_1 \geq \cdots \geq a_r$ required of a partition is not imposed. Let $c_n = |C(n)|$. Clearly $P(n) \subset C(n)$, and so $p_n \leq c_n$.

The number of compositions $(a_1, \ldots, a_r)$ of n having length r is $\binom{n-1}{r-1}$. This can be seen by imagining n balls in a row. There is a space between the first and the second, the second and the third, and so on, until the last space, between the $n-1$-st and the n-th ball. If we choose $r-1$ of these spaces, which we can in $\binom{n-1}{r-1}$ ways, then we obtain a composition of n of length r. Moreover, every such composition can be obtained in this way. For example, the selection of the three spaces indicated by the vertical lines:

$$\bullet\,\bullet\,|\,\bullet\,\bullet\,\bullet\,|\,\bullet\,|\,\bullet\,\bullet\,\bullet\,\bullet$$

corresponds to the composition $(2,3,1,4)$ of 10 having length 4.

Hence by the Binomial Theorem, the total number of compositions of n is

$$\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{n-1}=(1+1)^{n-1}=2^{n-1}.$$

Remembering that $p_k \le c_k$ for each k, we therefore get

$$p_1+\cdots+p_n \le c_1+\cdots+c_n = 2^0+\cdots+2^{n-1}=2^n-1<2^n.$$

7. In the cycle decomposition of a permutation, the cycles are disjoint, and so we can have at most one cycle of length greater than $n/2$. Let $k>n/2$. To obtain a permutation with a cycle of length k, we pick out the points to be in the k-cycle (in $\binom{n}{k}$ ways), make them into a k-cycle (in $(k-1)!$ ways), and permute the remaining $n-k$ points arbitrarily ($(n-k)!$ possibilities). So the set C_k of permutations having a cycle of length k has $\binom{n}{k}(k-1)!(n-k)!$ elements.

For $k>n/2$, i.e., for $k \ge \lfloor n/2\rfloor+1$, the sets C_k are disjoint. Hence, writing $m=\lfloor n/2\rfloor$, the proportion of permutations with a cycle of length greater than $n/2$ is

$$\begin{aligned}
p_n &= \frac{1}{n!}\sum_{k=m+1}^{n}|C_k| \\
&= \frac{1}{n!}\sum_{k=m+1}^{n}\binom{n}{k}(k-1)!(n-k)! \\
&= \sum_{k=m+1}^{n}\frac{1}{k} && (1)\\
&= H(n)-H(m) && (2)
\end{aligned}$$

where $H(n)=\sum_{k=1}^{n}1/k$. In (1), the sum has $n-m\ge n/2$ terms, each at least $1/n$. So we get $p_n \ge 1/2$.

We use the well-known result

$$H(n)=\ln n+\gamma+\epsilon_n,$$

where γ is Euler's constant and $\epsilon_n\to 0$ as $n\to\infty$, together with (2), to get the limiting behaviour of p_n. Recall that $m=\lfloor n/2\rfloor$ is either $n/2$ or $(n-1)/2$ according as n is even or odd. So $\ln n-\ln m=\ln(n/m)$ equals $\ln 2+\delta_n$, where δ_n is either 0 or $\ln(n/(n-1))$. Thus $\delta_n\to 0$ as $n\to\infty$. Hence

$$\begin{aligned}
p_n &= H(n)-H(m)\\
&= \ln n-\ln m+\epsilon_n-\epsilon_m\\
&= \ln 2+\delta_n+\epsilon_n-\epsilon_m\\
&\to \ln 2 \quad \text{as } n\to\infty.
\end{aligned}$$

8. Given n numbers, not necessarily distinct, we can form a set consisting of these numbers, but keeping track of the multiplicities of each of the numbers. We shall refer to such an object as a *multiset* of n numbers or of *size* n. We refer to the *sum* of the elements of a multiset as the ordinary sum of its elements, adding each element as often as its multiplicity. The statement to be proved is this:

$P(n)$: Given any multiset of $2n-1$ integers, there is a sub-multiset of size n for which the sum of the elements is divisible by n.

Note, incidentally, that the result is not true if $2n-1$ is replaced by $2n-2$, since the multiset consisting of $n-1$ 0's and $n-1$ 1's gives a counterexample.

First we show that $P(a)$ and $P(b)$ together imply $P(ab)$. So assume we are given $2ab-1$ integers. Since $2ab-1 \geq 2a-1$, we can find a of them whose sum is a multiple of a. Remove these a elements, and if we still have at least $2a-1$ left, repeat the argument, i.e., take out another a elements with sum divisible by a. We can clearly do this $2b-1$ times. So we get $2b-1$ multisets of size a, the sums of which are $s_1a, s_2a, \ldots, s_{2b-1}a$. By $P(b)$ we can choose b of the numbers $s_1, s_2, \ldots, s_{2b-1}$ with sum divisible by b. Then the union of the corresponding multisets of size a is a multiset of size ab with sum divisible by ab.

So we are reduced to proving $P(n)$ for n prime. Let $\mathbb{F}_p$ denote the set $\{0, 1, \ldots, p-1\}$, a field when addition and multiplication are taken mod p. We use induction on k to prove the following statement $Q(k)$ for all k from 2 to p (inclusive):

$Q(k)$: Given $2k-1$ elements of $\mathbb{F}_p$ such that no element is repeated k or more times, we can find k sub-multisets of size k giving k distinct sums.

If we can prove this, then $Q(p)$ says that given $2p-1$ integers mod p with no element repeated p or more times, then for each r in $\mathbb{F}_p$ one can find p of these elements with sum r. In particular, we can find p of them with sum 0. If some element is repeated p or more times we can obviously find p of them with sum 0. So $Q(p)$ implies $P(p)$.

$Q(2)$ says that given 3 distinct elements a, b, c there are two sets of size 2 with distinct sums. This is easy: $\{a, b\}$ and $\{a, c\}$ will do.

Assume that $2 \leq k \leq p-1$ and that $Q(k)$ holds, and we are given $2k+1$ elements of $\mathbb{F}_p$ with nothing repeated $k+1$ or more times. Let a be an element with maximal multiplicity in our given multiset S of size $2k+1$, and b an element different from a with multiplicity as large as possible. Remove a and b, leaving a multiset S' with $2k-1$ elements. Then a and b have multiplicity less than k in S', and if some element $c \neq a, b$ of S' has multiplicity k or more, then a and b have multiplicity at least k in the original multiset S, giving at least $3k$ elements, whereas in fact there are only $2k+1$. So in S' no element occurs with multiplicity k or more. By $Q(k)$, there are k submultisets $S_1, S_2, \ldots, S_k$, of S' of size k, with k different sums. Now the sets $S_1 \cup \{a\}, S_2 \cup \{a\}, \ldots, S_k \cup \{a\}$ have size $k+1$ and give k distinct sums. The same is true for the sets $S_i \cup \{b\}$. It suffices to show that one of the latter has sum different from all of the former. The only alternative is that the sums of the $S_j \cup \{a\}$ are a permutation of the sums of the $S_i \cup \{b\}$. Then the sum of all the former equals the sum of all the latter, and cancelling the common terms then gives $ka = kb$. But k is one of $2, 3, \ldots, p-1$, and so $\gcd(k, p) = 1$. So $a = b$, a contradiction.

9. Write N for the set $\{1, \ldots, n\}$. We show that with respect to the basis $\{e_S : S \subset N\}$, the linear transformation $Z = XY - YX$ is diagonal, with $Ze_S = (2|S| - n)e_S$ for each $S \subset N$.

We first deal with the special cases $S = \emptyset$ and $S = N$. Now $Xe_\emptyset = \sum_{i \in N} e_{\{i\}}$ and $Ye_\emptyset = 0$. Also, $Ye_{\{i\}} = e_\emptyset$ for each $i \in N$. Hence

$$Ze_\emptyset = X(Ye_\emptyset) - Y(Xe_\emptyset) = X(0) - Y\Big(\sum_{i \in N} e_{\{i\}}\Big) = 0 - \sum_{i \in N} Y(e_{\{i\}}) = -ne_\emptyset.$$

Next, $Xe_N = 0$ and $Ye_N = \sum_{i \in N} e_{N \setminus \{i\}}$, and $Xe_{N \setminus \{i\}} = e_N$ for each $i \in N$. Hence

$$Ze_N = X(Ye_N) - Y(Xe_N) = X\Big(\sum_{i \in N} e_{N \setminus \{i\}}\Big) - Y(0) = \sum_{i \in N} X(e_{N \setminus \{i\}}) = ne_N.$$

Now suppose that $\emptyset \subsetneqq S \subsetneqq N$. Let T denote the complement $N \setminus S$ of S in N. Then $Xe_S = \sum_{t \in T} e_{S \cup \{t\}}$ and $Ye_S = \sum_{s \in S} e_{S \setminus \{s\}}$. For each $s \in S$, $X\big(e_{S \setminus \{s\}}\big)$ is the sum of e_S and the basis vectors $e_{(S \setminus \{s\}) \cup \{t\}}$, $t \in T$. For each $t \in T$, $Y\big(e_{S \cup \{t\}}\big)$ is the sum of e_S and the basis vectors $e_{(S \cup \{t\}) \setminus \{s\}}$, $s \in S$. Notice that $(S \cup \{t\}) \setminus \{s\} = (S \setminus \{s\}) \cup \{t\} = S_{s,t}$, say, for all $s \in S$ and $t \in T$. Moreover, the $|S| \times |T|$ sets $S_{s,t}$ obtained by varying $s \in S$ and $t \in T$ are all distinct. Hence

$$XYe_S = \sum_{s \in S} X\big(e_{S \setminus \{s\}}\big) = \sum_{s \in S} \Big(e_S + \sum_{t \in T} e_{(S \setminus \{s\}) \cup \{t\}}\Big) = |S| e_S + \sum_{s \in S, t \in T} e_{S_{s,t}},$$

and

$$YXe_S = \sum_{t \in T} Y\big(e_{S \cup \{t\}}\big) = \sum_{t \in T} \Big(e_S + \sum_{s \in S} e_{(S \cup \{t\}) \setminus \{s\}}\Big) = |T| e_S + \sum_{s \in S, t \in T} e_{S_{s,t}}.$$

Subtracting, we get

$$(XY - YX)e_S = (|S| - |T|)e_S = (2|S| - n)e_S.$$

10. Given fractions a/b and e/f, where $0 < b, e, f \in \mathbb{N}$ and $0 \le a \in \mathbb{N}$, the condition $be - af = 1$ implies that $af < be$, so that $a/b < e/f$. We call the interval $I = [a/b, e/f]$ *good* in this case. Notice that the condition $be - af = 1$ also implies that $\gcd(a, b) = 1 = \gcd(e, f)$. Given a good interval I with left hand endpoint $\ell(I) = a/b$ and right hand endpoint $r(I) = e/f$, we call the fraction $(a + e)/(b + f)$ the *median* $m(I)$ of I. Notice that

$$b(a + e) - a(b + f) = (b + f)e - (a + e)f = be - af = 1,$$

and so $a/b < (a + e)/(b + f) < e/f$, $\gcd(a + e, b + f) = 1$, and so both $I' = [\ell(I), m(I)]$ and $I'' = [m(I), r(I)]$ are good.

We start from the good interval $I_{0,1} = [0/1, 1/1]$, and form $I_{1,1} = I'_{0,1} = [0/1, 1/2]$ and $I_{1,2} = I''_{0,1} = [1/2, 1/1]$. Then we form $I_{2,1} = I'_{1,1} = [0/1, 1/3]$, $I_{2,2} = I''_{1,1} = [1/3, 1/2]$, $I_{2,3} = I'_{1,2} = [1/2, 2/3]$ and $I_{2,4} = I''_{1,2} = [2/3, 1/1]$. Continuing, after n steps we have 2^n intervals $I_{n,j}$, $j = 1, \ldots, 2^n$, where $I_{n,2j-1} = I'_{n-1,j}$ and $I_{n,2j} = I''_{n-1,j}$ for $j = 1, \ldots, 2^{n-1}$. This divides $[0, 1]$ into 2^n non-overlapping intervals, with $\ell(I_{n,j}) = r(I_{n,j-1})$ for $j = 2, \ldots, 2^n$. So, given, $n \in \mathbb{N}$, each $x \in [0, 1]$ belongs to $I_{n,j}$ for some $j \in \{1, \ldots, 2^n\}$, and x belongs to two $I_{n,j}$'s if $x \neq 0, 1$ is an endpoint of an $I_{n,j}$. Notice that the two endpoints of a good interval I are endpoints of both I' and I''.

Given a rational number $q \in (0,1)$, we next show that q is the median of at most one of the intervals $I_{n,j}$. For q cannot be the median of both $I_{n,j}$ and $I_{n,k}$, where $j \neq k$, because the median of a good interval is interior to that interval, and the two intervals $I_{n,j}$ and $I_{n,k}$ can intersect in at most an endpoint. Nor can q be the median of both $I_{m,j}$ and $I_{n,k}$, where $m \neq n$. For if $m < n$, say, then q being the median of $I_{m,j}$ implies that q is an endpoint of $I_{m+1,2j-1}$ (and of $I_{m+1,2j}$), and so for all $m' > m$ q remains an endpoint of some $I_{m',i}$. In particular, q is an endpoint of an $I_{n,i}$ since $n > m$, and so cannot also be the median of $I_{n,k}$.

Given a rational number $q \in (0,1)$, it remains to show that q is the median of some $I_{n,j}$.

Case (a). Suppose firstly that q is an endpoint of some $I_{n,j}$. Let q be an endpoint of $I_{n,j}$, with n minimal. Then $n \neq 0$ because $q \neq 0, 1$. If q is the left endpoint of $I_{n,j}$, and j is odd, then q is also the left endpoint of $I_{n-1,(j+1)/2}$, contrary to the minimality of n. Similarly, if q is the right endpoint of $I_{n,j}$ and j is even, then q is also the right endpoint of $I_{n-1,j/2}$, again a contradiction. So either q is the left endpoint of $I_{n,j}$, with j even, in which case q is the median of $I_{n-1,j/2}$, or q is the right endpoint of $I_{n,j}$, with j odd, in which case q is the median of $I_{n-1,(j+1)/2}$.

Case (b). Suppose finally that $q \in (0,1)$ is rational, and not the endpoint of any $I_{n,j}$. Hence for each $n \in \mathbb{N}$, there is exactly one $j_n \in \{1, \ldots, 2^n\}$ such that $q \in I_{n,j_n}$. Write I_n instead of I_{n,j_n}. The intervals I_n are all distinct, because each $I_{n+1,j}$ is a proper subset of an $I_{n,k}$. But if $q = c/d$, if $I = [a/b, e/f]$ is good, and if $q \in (a/b, e/f)$, then $b + f \leq d$. To see this, notice that $ad < bc$, and so $bc - ad \geq 1$, so that $fbc - fad = f(bc - ad) \geq f$. Also, $cf < de$, and so $de - cf \geq 1$, so that $bde - fbc = b(de - cf) \geq b$. thus

$$d = d(be - af) = (dbe - fbc) + (fbc - daf) \geq b + d.$$

The condition $b + f \leq d$ shows that the denominators of the endpoints of any good I containing q are both bounded by d. For $I \subset [0,1]$, the numerators of the endpoints of I are therefore also bounded by d. Hence there are only finitely many good $I \subset [0,1]$ which can contain q. So it is impossible that $q \in I_n$ for $n = 1, 2, \ldots$. So Case (b) cannot happen.