Endpoint estimates for commutators with respect to the fractional integral operators on Orlicz-Morrey spaces

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1. Background

- For $0 < \alpha < n$, define

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y, \quad x \in \mathbb{R}^n.$$

- Sobolev's inequality: For all $f \in L^p(\mathbb{R}^n)$,

$$p > 1, \; \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \Longrightarrow \quad \|I_\alpha f\|_{L^s} \lesssim \|f\|_{L^p}.$$

Definition

Let $0 < \alpha < n$, and let b be a measurable function on $\mathbb{R}^n$.

- *Commutators: For $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$,*

$$[b, I_\alpha]f(x) \equiv b(x) I_\alpha f(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{|x-y|^{n-\alpha}} f(y) \, \mathrm{d}y.$$

- $\mathrm{BMO}(\mathbb{R}^n)$:

$$\|b\|_{\mathrm{BMO}} \equiv \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, \mathrm{d}x, \quad b_Q \equiv \frac{1}{|Q|} \int_Q b(x) \, \mathrm{d}x.$$

Theorem 1 (Chanillo, 1982)

Let $0 < \alpha < n$ *and* $1 < p < s < \infty$*. Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then

$$b \in \mathrm{BMO}(\mathbb{R}^n) \iff \|[b, I_\alpha]f\|_{L^s} \lesssim \|f\|_{L^p} \text{ for all } f \in C_c^\infty(\mathbb{R}^n).$$

※ $C_c^\infty(\mathbb{R}^n) \underset{\text{dense}}{\subset} L^p(\mathbb{R}^n)$

$\rightsquigarrow$ Commutators are well defined on $L^p(\mathbb{R}^n)$.

Taking

$$f(x) = \chi_{[0,1]}(x), \quad b(x) = \log(1+x)\chi_{(1,\infty)}(x),$$

then we can see that $[b, I_\alpha]$ is not weak-$\left(1, \dfrac{n}{n-\alpha}\right)$ bounded (cf. Cruz-Uribe and Fiorenza, 2003).

Endpoint estimates

We may use Orlicz spaces.

Theorem 2 (Cruz-Uribe and Fiorenza, 2003)

Let $0 < \alpha < n$. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$, *then*

$$|\{x \in \mathbb{R}^n : |[b, I_\alpha]f(x)| > 1\}| \lesssim \Psi\left(\int_{\mathbb{R}^n} \Phi(\|b\|_{\mathrm{BMO}}|f(x)|)\,\mathrm{d}x\right),$$

where we put

$$\Phi(t) \equiv t\log(3+t), \quad \Psi(t) \equiv \left(t\log\left(3+t^{\frac{\alpha}{n}}\right)\right)^{\frac{n}{n-\alpha}}.$$

※ It follows that

$$\|[b, I_\alpha]f\|_{\mathrm{W}L^{\Phi_1}} \lesssim \|f\|_{L^\Phi}$$

holds for some Young function Φ_1.

2. Function spaces

Morrey and weak Morrey spaces

Definition

Let $0 < q \leq p < \infty$.

- *Morrey space* $\mathcal{M}^p_q(\mathbb{R}^n)$:

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{Q:\mathrm{cube}} |Q|^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}}.$$

- *Weak Morrey space* $\mathrm{W}\mathcal{M}^p_q(\mathbb{R}^n)$:

$$\|f\|_{\mathrm{W}\mathcal{M}^p_q} \equiv \sup_{t>0} \left\| t\chi_{\{x\in\mathbb{R}^n \,:\, |f(x)|>t\}} \right\|_{\mathcal{M}^p_q}.$$

- $\mathcal{M}^p_p(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad \mathrm{W}\mathcal{M}^p_p(\mathbb{R}^n) = \mathrm{W}L^p(\mathbb{R}^n)$.
- $0 < q_2 \le q_1 < p < \infty$

$$\Longrightarrow \quad \mathcal{M}^p_{q_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2}(\mathbb{R}^n), \quad \mathrm{W}\mathcal{M}^p_{q_1}(\mathbb{R}^n) \hookrightarrow \mathrm{W}\mathcal{M}^p_{q_2}(\mathbb{R}^n).$$

- $0 < q < p < \infty \quad \Longrightarrow \quad \mathrm{W}L^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$.
- $0 < q < p < \infty \quad \Longrightarrow \quad C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_q(\mathbb{R}^n)$.

Theorem 3 (Adams' theorem; 1975, 1987)

Let $0 < \alpha < n$, $1 < q \le p < \infty$ and $1 < t \le s < \infty$. Assume that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

(1) $\|I_\alpha f\|_{\mathcal{M}^s_t} \lesssim \|f\|_{\mathcal{M}^p_q}$.

(2) $\|I_\alpha f\|_{\mathrm{W}\mathcal{M}^s_{t_0}} \lesssim \|f\|_{\mathcal{M}^p_1} \quad \left(\frac{1}{p} = \frac{t_0}{s}\right)$.

Orlicz and weak Orlicz spaces

Definition

(1) *A function* $\Phi : (0,\infty) \to (0,\infty)$ *is called a Young function, if it satisfies the following properties*:

- $\lim_{t\to 0^+} \Phi(t) = 0.$
- Φ *is convex.*

(2) *Orlicz space* $L^\Phi(\mathbb{R}^n)$:

$$\|f\|_{L^\Phi} \equiv \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \le 1\right\}.$$

(3) *Weak Orlicz space* $\mathrm{W}L^\Phi(\mathbb{R}^n)$:

$$\begin{aligned}\|f\|_{\mathrm{W}L^\Phi} &\equiv \sup_{t>0} \left\| t\chi_{\{x\in\mathbb{R}^n \,:\, |f(x)|>t\}} \right\|_{L^\Phi} \\ &= \inf\left\{\lambda > 0 \,:\, \sup_{t>0} \Phi(t) \left|\left\{x \in \mathbb{R}^n \,:\, \frac{|f(x)|}{\lambda} > t\right\}\right| \le 1\right\}.\end{aligned}$$

※ When $\Phi(t) = t\log(3+t)$, we write $L^\Phi(\mathbb{R}^n) = L\log L(\mathbb{R}^n)$.

- Hardy-Littlewood maximal operaotor M:

$$Mf(x) \equiv \sup_{Q:\text{cube}} \frac{\chi_Q(x)}{|Q|} \int_Q |f(y)|\,\mathrm{d}y.$$

Definition

For a Young function Φ,

$$\Phi \in \nabla_2 \quad \underset{\text{def}}{\Longleftrightarrow} \quad {}^{\exists}k > 1 \text{ s.t. } \Phi(r) \le \frac{1}{2k}\Phi(kr) \text{ for } r > 0.$$

The $\mathrm{W}L^{\Phi}$-boundedness of M is given as follows:

Theorem 4 (Kawasumi, Nakai and Shi, 2021)

If $\Phi \in \nabla_2$, *then there exists* $C_\Phi > 0$ *such that*

$$\sup_{t>0} \Phi(t)\,|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le \sup_{t>0} \Phi(t)\,|\{x \in \mathbb{R}^n : C_\Phi|f(x)| > t\}|.$$

Orlicz average and Orlicz-Morrey spaces

Definition (Sawano, Sugano and Tanaka, 2012)

Let $1 < p < \infty$.

(1) *Orlicz average: For each cube Q, define*

$$\|f\|_{L\log L,Q} \equiv \inf\left\{\lambda > 0 \, : \, \frac{1}{|Q|}\int_Q \frac{|f(x)|}{\lambda}\log\left(3+\frac{|f(x)|}{\lambda}\right)\mathrm{d}x \le 1\right\}.$$

(2) *Orlicz-Morrey space* $\mathcal{M}^p_{L\log L}(\mathbb{R}^n)$:

$$\|f\|_{\mathcal{M}^p_{L\log L}} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}}\|f\|_{L\log L,Q}.$$

- $1 < q \le p < \infty$

$$\Longrightarrow \quad \mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{L\log L}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_1(\mathbb{R}^n).$$

- $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}^p_{L\log L}(\mathbb{R}^n)$.

3. Main results

Commutators on Morrey spaces

Theorem 5

Let $0 < \alpha < n$, $1 < q \le p < \infty$ *and* $1 < t \le s < \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then

$$b \in \mathrm{BMO}(\mathbb{R}^n) \quad \Longleftrightarrow \quad \|[b, I_\alpha]f\|_{\mathcal{M}^s_t} \lesssim \|f\|_{\mathcal{M}^p_q}.$$

- ($\Longrightarrow$): Di Fazio and Ragusa, 1991.
- ($\Longleftarrow$): Shirai, 2006.

Theorem 6 (Sawano and Hakim, 2021)

Let $0 < \alpha < n$, $1 < q \le p < \infty$ *and* $1 < t \le s < \infty$. *Assume that*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{q}{p} = \frac{t}{s}.$$

Then

$$b \in \mathrm{BMO}(\mathbb{R}^n) \iff \|[b, I_\alpha]f\|_{\mathcal{M}^s_t} \lesssim \|f\|_{[\mathcal{M}^p_q, \mathcal{M}^p_{L\log L}]^{\alpha p/n}}.$$

- $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow [\mathcal{M}^p_q(\mathbb{R}^n), \mathcal{M}^p_{L\log L}(\mathbb{R}^n)]^{\alpha p/n} \hookrightarrow \mathcal{M}^p_{L\log L}(\mathbb{R}^n)$.

Main theorem (H.)

Let $0 < \alpha < n$, $1 < p < \infty$ and $1 < t < s < \infty$. Assume that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{p} = \frac{t}{s}.$$

Then the following assertions are equivalent:

(1) $b \in \mathrm{BMO}(\mathbb{R}^n)$.

(2) For all $f \in \mathcal{M}^p_{L\log L}(\mathbb{R}^n)$,

- $[b, I_\alpha]f$ is well defined.
- $\|[b, I_\alpha]f\|_{\mathrm{w}\mathcal{M}^s_t} \lesssim \|f\|_{\mathcal{M}^p_{L\log L}}$.

The proof of the statement (2) $\Longrightarrow$ (1) is similar to the ideas from Janson in 1978.

To prove the statement (1) $\Longrightarrow$ (2), we give the estimate for

$$|b, I_\alpha| f(x) \equiv \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|}{|x - y|^{n-\alpha}} f(y) \, \mathrm{d}y.$$

Main theorem (H.)

Let $0 < \alpha < n$, $1 < t < s < \infty$ and $1 < p < \infty$. Assume that

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{p} = \frac{t}{s}.$$

Then

$$b \in \mathrm{BMO}(\mathbb{R}^n) \quad \Longrightarrow \quad \| |b, I_\alpha| f \|_{\mathrm{W}\mathcal{M}^s_t} \lesssim \|f\|_{\mathcal{M}^p_{L \log L}}.$$

4. Proof of the main theorem

Fix a cube Q. We decompose

$$f = f\chi_{2Q} + f\chi_{\mathbb{R}^n\setminus 2Q} =: f_1 + f_2.$$

We omit the estimate for f_2 in this talk.

About the estimate for f_1, the proof is constructed by the following claims:

(i) $(|b, I_\alpha|g)^\sharp(x) \lesssim \|b\|_{\mathrm{BMO}}\left(M((I_\alpha g)^\eta)(x)^{\frac{1}{\eta}} + M_{\alpha, L\log L}g(x)\right)$.

(ii) $\|g\|_{\mathrm{w}\mathcal{M}^s_t} \lesssim \|g^\sharp\|_{\mathrm{w}\mathcal{M}^s_t}$.

(iii) $\|M_{\alpha, L\log L}g\|_{\mathrm{w}\mathcal{M}^s_t} \lesssim \|g\|_{\mathcal{M}^p_{L\log L}}$.

To prove (ii), we referred the following paper:

S. Nakamura and Y. Sawano, The singular integral operator and its commutator on weighted Morrey spaces, Collect. Math. 68 (2017), no. 2, 145–174.

- Fefferman-Stein sharp maximal function:

$$f^\sharp(x) \equiv \sup_{Q:\text{cube}} \frac{\chi_Q(x)}{|Q|}\int_Q |f(y) - f_Q|\,\mathrm{d}y.$$

- $M_{\alpha, L\log L}f(x) \equiv \sup_{Q:\text{cube}} \chi_Q(x)|Q|^{\frac{\alpha}{n}}\|f\|_{L\log L, Q}$.

Theorem 7 (cf. Cruz-Uribe and Fiorenza, 2003)

Let $0 < \alpha < n$, $1 < p < \infty$ *and* $\eta > 1$. *If*

$$b \in \mathrm{BMO}(\mathbb{R}^n), \quad \frac{1}{p} - \frac{\alpha}{n} > 0,$$

then

$$(|b, I_\alpha|f)^\sharp(x) \lesssim \|b\|_{\mathrm{BMO}}\left(M((I_\alpha f)^\eta)(x)^{\frac{1}{\eta}} + M_{\alpha, L\log L}f(x)\right)$$

for all nonnegative functions $f \in \mathcal{M}^p_{L\log L}(\mathbb{R}^n)$.

(ii) $\|g\|_{\mathrm{w}\mathcal{M}^s_t} \lesssim \|g^\sharp\|_{\mathrm{w}\mathcal{M}^s_t}$

Definition

(1) *Rearrangement*:

$$f^*(t) \equiv \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \le t\}, \quad t > 0.$$

(2) *Local mean oscillation*: *For a cube Q,*

$$\omega_\lambda(f; Q) \equiv \inf_{c \in \mathbb{C}} ((f - c)\chi_Q)^*(\lambda|Q|), \quad 0 < \lambda \ll 1.$$

(3) *Sharp maximal function*:

$$M^{\sharp,d}_\lambda f(x) \equiv \sup_{Q:\text{cube}} \omega_\lambda(f; Q)\chi_Q(x), \quad x \in \mathbb{R}^n.$$

Proposition 8 (Jawerth and Torchinsky, 1985)

$$M \circ M^{\sharp,d}_\lambda f(x) \sim f^\sharp(x).$$

The median $m_f(Q)$ is defined by a real number satisfying

$$|\{x \in Q : |f(x)| > m_f(Q)\}|, \ |\{x \in Q : |f(x)| < m_f(Q)\}| \leq \frac{|Q|}{2}.$$

Theorem 9 (cf. Nakamura and Sawano, 2017)

Let $0 < q \leq p < \infty$. *Then*

$$m_f(2^\ell Q) \to 0 \ (\ell \to \infty, \ ^\forall Q) \quad \Longrightarrow \quad \|f\|_{\mathrm{W}\mathcal{M}^p_q} \sim \left\| M^{\sharp,d}_\lambda f \right\|_{\mathrm{W}\mathcal{M}^p_q}.$$

To prove this theorem, we may use the following proposition.

We say that the family $\{Q_j^k\}_{k\in\mathbb{N}_0, j\in J_k} \subset \mathcal{Q}(\mathbb{R}^n)$ is a sparse family if the following properties hold: for each $k \in \mathbb{N}_0$,

(1) the cubes $\{Q_j^k\}_{j\in J_k}$ are disjoint;

(2) if $\Omega_k \equiv \bigcup_{j\in J_k} Q_j^k$, then $\Omega_{k+1} \subset \Omega_k$;

(3) $2|\Omega_{k+1} \cap Q_j^k| \le |Q_j^k|$ for all $j \in J_k$.

Proposition 10 (Lerner, 2013)

There exists a sparse family of $\{Q_j^k\}_{k\in\mathbb{N}_0, J_k} \subset \mathcal{D}(Q_0)$ such that

$$|f - m_f(Q_0)| \lesssim M_\lambda^{\sharp,d} f + \sum_{k\in\mathbb{N}_0} \sum_{j\in J_k} \omega_\lambda(f; Q_j^k)\chi_{Q_j^k}.$$

- $M_{\alpha,L\log L}f(x) \equiv \sup_{Q\in\mathcal{Q}} \chi_Q(x)\ell(Q)^\alpha\|f\|_{L\log L,Q}$.

Theorem 11

Let $0 \le \alpha < n$, $1 < t < s < \infty$ *and* $1 < p < \infty$. *If*

$$\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{1}{p} = \frac{t}{s},$$

then

$$\|M_{\alpha,L\log L}f\|_{\mathrm{w}\mathcal{M}^s_t} \sim \|f\|_{\mathcal{M}^p_{L\log L}}.$$

To use Theorem 9, we have to verify the statement

$$m_{|b,I_\alpha|f_1}(2^\ell Q_0) \to 0 \quad (\ell \to \infty)$$

for all cubes Q_0.

Lemma 12 (cf. Nakamura and Sawano, 2017)

Let f be a measurable function. If $Mf \in \mathrm{W}L^\Phi(\mathbb{R}^n)$ for some Young function Φ, then

$$\lim_{\ell\to\infty} m_f(2^\ell Q) = 0$$

for all cubes Q.

Define

$$\Phi(t) \equiv t\log(3+t), \quad \Psi(t) \equiv \left(t\log\left(3+t^{\frac{\alpha}{n}}\right)\right)^{\frac{n}{n-\alpha}},$$

$$\Phi_0(t) \equiv \left(\frac{t}{\log^2(3+1/t)}\right)^{\frac{n}{n-\alpha}}.$$

Lemma 13

Let $0 < \alpha < n$.

(1) $\Psi \circ \Phi(t) \lesssim \left(t\log^2(3+t)\right)^{n/(n-\alpha)}$.

(2) ${}^{\exists}\Phi_1 \in \nabla_2$ *s.t.* $\Phi_1(t) \lesssim \Phi_0(t)$.

$$\begin{array}{ccccccc} L^{\Phi}(\mathbb{R}^n) & \xrightarrow{|b,I_\alpha|} & \mathrm{W}L^{\Phi_0}(\mathbb{R}^n) & \hookrightarrow & \mathrm{W}L^{\Phi_1}(\mathbb{R}^n) & \xrightarrow{M} & \mathrm{W}L^{\Phi_1}(\mathbb{R}^n) \\ \Cup & & \Cup & & \Cup & & \Cup \\ f_1 & \mapsto & |b,I_\alpha|f_1 & \mapsto & |b,I_\alpha|f_1 & \mapsto & M(|b,I_\alpha|f_1). \end{array}$$

Hence, we can use Lemma 12... □

References

[1] S. Chanillo, A note on commutators, Indiana Univ. Math. J. **31**(1), 7–16 (1982).

[2] D. Cruz-Uribe and A. Fiorenza, Endpoint estimates and weighted norm inequalities for commutators of fractional integrals, Publ. Mat. 47 (2003), no. 1, 103–131.

[3] G. Di Fazio and M.A.G. Ragusa, Commutators and Morrey spaces, Boll. Un. Mat. Ital. **A (7) 5** (1991), no. 3, 323–332.

[4] S. Janson, Mean oscillation and commutators of singular integral operators, Ark. Mat. 16 (1978), no. 2, 263–270.

[5] B. Jawerth and A. Torchinsky, Local sharp maximal functions, J. Approx. Theory 43 (1985), no. 3, 231–270.

[6] R. Kawasumi, E. Nakai and M. Shi, Characterization of the boundedness of generalized fractional integral and maximal operators on Orlicz-Morrey and weak Orlicz-Morrey spaces, arXiv:2107.10553.

[7] S. Nakamura and Y. Sawano, The singular integral operator and its commutator on weighted Morrey spaces, Collect. Math. 68 (2017), no. 2, 145–174.

[8] Y. Sawano and D. I. Hakim, Complex interpolation and commutators acting on Morrey spaces, Rom. J. Math. Comput. Sci. 11 (2021), no. 1, 10–24.

[9] Y. Sawano, S. Sugano and H. Tanaka, Orlicz-Morrey spaces and fractional operators, Potential Anal. 36 (2012), no. 4, 517–556.

[10] S. Shirai, Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces, Hokkaido Math. J. 35 (2006), no. 3, 683–696.