Spreading rate for the Fisher-KPP nonlocal diffusion equation with free boundary

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1. Brief Review of Results with Local Diffusion

(a) The classical Fisher-KPP equation

Starting from the pioneering works of **Fisher** (1937) and **KPP** (Kolmogorov-Petrovski-Piskunov, 1937), the Cauchy problem

$$(1) \qquad \begin{cases} U_t = dU_{xx} + f(U), & t > 0,\ x \in \mathbb{R}, \\ U(0,x) = U_0(x), & x \in \mathbb{R}, \end{cases}$$

has been widely used to describe the spreading of a population with density $U(t,x)$ at time t and space location x, where the initial population $U_0(x)$ is a nonnegative function with compact support, and the growth term $f(U)$ is usually a C^1 function satisfying $f(0) = 0$. The diffusion term

$$dU_{xx}$$

is used to describe the dispersal of the population through local random walk.

A striking feature of (1), with

$$f(U) = U(1-U) \text{ (a prototype Fisher-KPP nonlinearity)},$$

is the following:

There exists a constant $c^* > 0$ such that

$$\begin{cases} \lim_{t\to\infty} U(t,x) = 1 & \text{uniformly in } |x| \le (c^* - \epsilon)t, \\ \lim_{t\to\infty} U(t,x) = 0 & \text{uniformly in } |x| \ge (c^* + \epsilon)t \end{cases}$$

for any small $\epsilon > 0$.

Interpretation: The population spreads into new space with (asymptotic) speed c^*.

Minimal speed of traveling waves: For any $c \geq c^* := 2\sqrt{d}$, (1) has a traveling wave solution with velocity c: $U(t,x) := V(ct-x)$, where V satisfies the ODE

$$dV'' - cV' + f(V) = 0, \; V(-\infty) = 0, \; V(+\infty) = 1.$$

There is no such solution for $c < c^*$.

Q: What is the spreading front determined by (1)?

$$\text{Population range } \Omega(t) := \{x : U(t,x) > 0\} = \mathbb{R} \text{ for } t > 0.$$

Ramification: Nominate a small $\delta > 0$ and regard the population range as

$$\Omega_\delta(t) := \{x : U(t,x) > \delta\},$$

which is a bounded set at any time $t > 0$. Let $[g_\delta(t), h_\delta(t)]$ be the smallest interval such that $\Omega_\delta(t) \subset [g_\delta(t), h_\delta(t)]$, then $x = g_\delta(t)$ and $x = h_\delta(t)$ could be viewed as the spreading fronts, which propagates to infinity with speed c^*:

$$\lim_{t\to\infty} \frac{g_\delta(t)}{-t} = \lim_{t\to\infty} \frac{h_\delta(t)}{t} = c^*.$$

(b) A free boundary problem

$$(2)\quad \begin{cases} u_t = du_{xx} + f(u) & \text{for } \; x \in (g(t), h(t)), \; t > 0 \\ u = 0, \quad \xi_t = -\mu u_x(t, \xi(t)) & \text{for } \; \xi(t) \equiv g(t) \text{ or } h(t), \\ u(0, x) = u_0(x) & \text{for } \; x \in (g(0), h(0)). \end{cases}$$

- Population range: $\Omega(t) := (g(t), h(t))$, $\Omega(0) = (g(0), h(0))$,
- Range boundary (free boundary): $\partial\Omega(t) = \{g(t), h(t)\}$,
- $f(u) = u(1-u)$ (for simplicity),
- Ω_0 bounded interval,
- $u_0 \in C^2(\overline{\Omega}_0)$, positive in Ω_0, $u_0|_{\partial\Omega_0} = 0$.

By **Du-Lin** (SIMA 2010), (2) has a unique classical solution (u, g, h) defined for all $t > 0$.

The free boundary condition can be deduced from suitable biological assumptions [**G. Bunting, Y. Du and K. Krakowski**, *Netw. Heterogeneous Media* **2012**].

Main features of (2):

- **Spreading-vanishing dichotomy**
  As $t \to \infty$, one of the following happens,

Spreading: $$\begin{cases} \lim_{t\to\infty} \Omega(t) = \mathbb{R}, \\ \lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } x, \end{cases}$$

Vanishing: $$\begin{cases} \lim_{t\to\infty} \Omega(t) = (g_\infty, h_\infty) \text{ is a finite interval}, \\ \lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0. \end{cases}$$

- **Spreading speed**
  If spreading happens, there exists $c_0 = c_0(\mu) > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = c_0.$$

- **Threshold result**
  There exists $\mu^* \geq 0$ (depending on u_0) such that spreading happens if and only if $\mu > \mu^*$.

- **Semi-wave**

  The spreading speed $c_0(\mu)$ of (2) is determined by the following result: *For any* $\mu > 0$ *the following system*

$$\begin{cases} dq'' - cq' + f(q) = 0, \ q > 0 \ \text{ in } (0, \infty), \\ q(0) = 0, \ q(\infty) = 1, q'(0) = \frac{c}{\mu} \end{cases}$$

  has a unique solution pair $(c, q) = (c_0, q_{c_0})$. *Moreover,*

$$c_0 = c_0(\mu) \textit{ increases to } c^* \textit{ as } \mu \to \infty.$$

  The function q_{c_0} is called a semi-wave with speed c_0.

- **Limiting problem as $\mu \to \infty$**

  If u *and* $\Omega(t)$ *in* (2) *are denoted by* u_μ *and* $\Omega_\mu(t)$, *respectively, then as* $\mu \to \infty$,

$$\Omega_\mu(t) \to \mathbb{R}(\forall t > 0), \ u_\mu \to U \textit{ in } C^{1,2}_{loc}((0, \infty) \times \mathbb{R}),$$

  where U *is the unique solution of* (1) *with* $U_0 = u_0$.

Therefore, (1) **may be viewed as the limiting problem of** (2) as $\mu \to \infty$.

The above results for (1) can be found in the classical work of **Aronson-Weinberger** (AdvMath1978), those for (2) can be found in **Du-Lin** (SIMA 2010), **Du-Guo** (JDE 2011), **Du-Lou** (JEMS 2015).

For both (1) and (2), further (much sharper) results have been obtained by many people, including successful extensions to

- high space dimensions,
- systems of equations,
- various heterogeneous environments.

Yet, in both (1) and (2), using du_{xx} (local diffusion) to describe the spatial dispersal of a population is not ideal in many situations, and replacing it by a nonlocal diffusion operator is more realistic.

The nonlocal version of (1) has been extensively investigated in the past 10-20 years, and fast progress is still being made. Research on the nonlocal version of (2) has just started.

2. The classical Fisher-KPP model with nonlocal diffusion

At any time $t > 0$, an individual at location x can jump to any other location y with probability $J(x-y)$. Under this assumption, the term du_{xx} should be replaced by

$$d\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy. \quad \text{(nonlocal diffusion operator)}$$

Thus a widely used nonlocal version of (1) is

$$(3)\quad \begin{cases} u_t = d\int_{\mathbb{R}} J(x-y)\big[u(t,y)-u(t,x)\big]\,dy + f(u), x\in\mathbb{R}, t>0, \\ u(0,x) = u_0(x) \text{ for } x\in\mathbb{R}. \end{cases}$$

(i) Usual assumptions on the convolution kernel $J(x)$:

$$J \in C(\mathbb{R}), \text{ is nonnegative and even, } \quad \int_{\mathbb{R}} J(x)dx = 1.$$

(ii) Thin-tailed and fat-tailed convolution kernel:

$J(x)$ is "thin-tailed" if

$$(\mathbf{J}_{\text{thin}}): \quad \int_0^\infty J(x)e^{\lambda x}dx < \infty \text{ for some } \lambda > 0.$$

Otherwise it is called "fat-tailed".

When the convolution kernel in (3) is thin-tailed, much of the basic theory for (1) carries over (by work of P. Bates, J. Coville, P. Fife, W. Shen, X. Wang, H. Weinberger, H. Yagisida, ...).

On the other hand, accelerated spreading happens when the kernel function is fat-tailed:

Weinberger (SIMA 1982): Let $u(t,x)$ be the solution of (3). Then $u(t,x)>0$ for $t>0$, $x\in\mathbb{R}$, and $\lim_{t\to\infty}u(t,x)=1$ locally uniformly for $x\in\mathbb{R}$. Moreover, for any given $\delta\in(0,1)$, if $[g_\delta(t),h_\delta(t)]$ is the smallest interval containing $\Omega_\delta(t):=\{x: u(t,x)>\delta\}$, then

$$\lim_{t\to\infty}\frac{h_\delta(t)}{t}=\lim_{t\to\infty}\frac{g_\delta(t)}{-t}=\begin{cases} c_*\in(0,\infty) & \text{if } J \text{ is thin-tailed},\\ \infty & \text{if } J \text{ is fat-tailed}^1.\end{cases}$$

J. Garnier (SIMA 2011): Examples of J are given such that $h_\delta(t)$ and $-g_\delta(t)$ behave like

$$\begin{cases} e^{\alpha t}\ (\alpha>0) & \text{when } J(x)\sim|x|^\sigma\ (\sigma<-2),\\ t^\beta\ (\beta>1) & \text{when } J(x)\sim e^{-|x|^{1/\beta}}.\end{cases}$$

And many further results along this line appeared in recent years.

[1] Weinberger & X.-Q. Zhao (Math. Bios. 2010)

(iii) The fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$:

- Convolution kernel of $(-\Delta)^s$:

$$k(|z|) = c_{N,s}|z|^{-(N+2s)}, \quad c_{N,s} := \frac{4^s\Gamma(\frac{N}{2}+s)}{\pi^{N/2}|\Gamma(s)|},$$

which is singluar at 0 and $\int_{\mathbb{R}^N} k(|z|)dz = \infty$. The convolution operator is understood as

$$\int_{\mathbb{R}^N} \frac{u(t,y)-u(t,x)}{|x-y|^{N+2s}}dy := \lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(0)} \frac{u(t,y)-u(t,x)}{|x-y|^{N+2s}}dy.$$

- Accelerated spreading speed [**Cabré-Roquejoffre**(CMP 2013)]:

  If $-du_{xx}$ *in* (1) *is replaced by* $(-\Delta u)^s$ *with* $s \in (0,1)$, *then as* $t \to \infty$, *for any small* $\epsilon > 0$,

$$\begin{cases} u(t,x) \to 0 \text{ uniformly in } \left\{|x| \geq e^{(\sigma_* + \epsilon)t}\right\}; \\ u(t,x) \to 1 \text{ uniformly in } \left\{|x| \leq e^{(\sigma_* - \epsilon)t}\right\}, \end{cases}$$

  where

$$\sigma_* := \frac{1}{N+2s}.$$

  So the spreading front propagates exponentially in time.

3. The 1-d free boundary model with nonlocal diffusion

$$
(4) \quad \begin{cases}
u_t = d\displaystyle\int_{\mathbb{R}} J(x-y)\big[u(t,y) - u(t,x)\big]\,dy + f(u), \; g(t) < x < h(t), \\
u(t,g(t)) = u(t,h(t)) = 0, \\
h'(t) = \mu \displaystyle\int_{g(t)}^{h(t)} \int_{h(t)}^{+\infty} J(x-y)u(t,x)\,dydx, \\
g'(t) = -\mu \displaystyle\int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)\,dydx, \\
u(0,x) = u_0(x), \; h(0) = -g(0) = h_0, \qquad x \in [-h_0, h_0],
\end{cases}
$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, which is always assumed to be identically 0 for $x \in \mathbb{R} \setminus [g(t), h(t)]$.

The initial function $u_0(x)$ satisfies $u_0 \in C([-h_0, h_0])$, and

$$u_0(-h_0) = u_0(h_0) = 0 \;\text{ and }\; u_0(x) > 0 \;\text{ in }\; (-h_0, h_0),$$

so $[-h_0, h_0]$ represents the initial population range of the species.

We assume that the kernel function $J : \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative, and has the properties

$$\textbf{(J):} \qquad J(0) > 0, \; \int_{\mathbb{R}} J(x)dx = 1, \; J(x) = J(-x), \; \sup_{\mathbb{R}} J < \infty.$$

As before, for simplicity, we take the special Fisher-KPP type nonlinearity

$$f(u) = u(1-u).$$

Note that

$$d\int_{\mathbb{R}} J(x-y)\left[u(t,y)-u(t,x)\right]dy = d\int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - du(t,x).$$

Meaning of the free boundary conditions

The total population mass moved out of the range $[g(t), h(t)]$ at time t through its right boundary $x = h(t)$ per unit time is given by

$$d \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t,x)dydx.$$

As we assume that $u(t,x) = 0$ for $x \notin [g(t), h(t)]$, this quantity of mass is lost in the spreading process of the species. We may call this quantity the outward flux at $x = h(t)$ and denote it by $J_h(t)$. Similarly we can define the outward flux at $x = g(t)$ by

$$J_g(t) := d \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t,x)dydx.$$

Then the free boundary conditions in (4) can be interpreted as assuming that the expanding rate of the front is proportional to the outward flux (by a factor μ/d):

$$g'(t) = -\mu J_g(t), \ h'(t) = \mu J_h(t).$$

Remarks:

- Problem (4) was first proposed in
  (i) Jiafeng Cao, Y. Du, Fang Li and Wan-Tong Li, **JFA 2019**,
  (ii) C. Cortázar, F. Quirós and N. Wolanski, **Interfaces Free Bound. 2019**.

  In (ii) the case $f(u) \equiv 0$ was considered. The free boundary conditions were proposed independently in these two papers.
- For a plant species, seeds carried across the range boundary may fail to establish due to numerous reasons, such as isolation from other members of the species causing poor or no pollination, or causing overwhelming attacks from enemy species. However, some of those not very far from the range boundary may survive, which results in the expansion of the population range. The free boundary condition here assumes that this survival rate is roughly a constant for a given species. For an animal species, a similar consideration can be applied.

Main results

(a) Spreading-vanishing dichotomy and criteria[2]

Theorem 1 (Existence and uniqueness:) Problem (4) has a unique solution (u, g, h) defined for all $t > 0$.

Theorem 2 (Spreading-vanishing dichotomy): Let (u, g, h) be the unique solution of (4). Then one of the following happens:

- Spreading $\begin{cases} \lim_{t\to+\infty}(g(t), h(t)) = \mathbb{R}, \\ \lim_{t\to+\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}, \end{cases}$
- Vanishing $\begin{cases} \lim_{t\to+\infty}(g(t), h(t)) = (g_\infty, h_\infty) \text{ is a finite interval}, \\ \lim_{t\to+\infty} \|u(t,\cdot)\|_{L^\infty([g(t),h(t)])} = 0. \end{cases}$

Theorem 3 (Spreading-vanishing criteria):

(α) If $d \le f'(0) = 1$, then spreading always happens.

(β) If $d > f'(0) = 1$, then there exists a unique $\ell^* > 0$ such that spreading always happens if $h_0 \ge \ell^*$; and for $h_0 \in (0, \ell^*)$, there exists a unique $\mu^* > 0$ so that spreading happens exactly when $\mu > \mu^*$.

[2] Cao-Du-Li-Li, JFA 2019

(b) Spreading speed[3]

We need to introduce a key condition on the kernel function, namely

$$\textbf{(J1):} \int_0^{\infty} xJ(x)\,dx < +\infty.$$

Theorem 4 (Spreading speed): Suppose **(J)** is satisfied, and spreading happens to the unique solution (u, g, h) of (4). Then the following conclusions hold.

- If **(J1)** is satisfied, then there exists a unique $c_0 > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = c_0. \quad \text{(linear spreading)}$$

- If **(J1)** does not hold, then

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{-t} = \infty. \quad \text{(accelerated spreading)}$$

[3] Y. Du, Fang Li and Maolin Zhou, accepted by JMPA (arXiv: 1909.03711)

The spreading speed c_0 is determined by semi-wave solutions to (4). These are pairs (c, ϕ) determined by the following system:

$$(5)\quad \begin{cases} d\int_{-\infty}^{0} J(x-y)\phi(y)dy - d\phi(x) + c\phi'(x) + f(\phi(x)) = 0, \ \ x<0, \\ \phi(-\infty) = 1, \quad \phi(0) = 0, \\ c = \mu \int_{-\infty}^{0} \int_{0}^{+\infty} J(x-y)\phi(x)dydx. \end{cases}$$

Theorem 5 (Semi-wave): Suppose **(J)** holds. Then (5) has a solution pair $(c, \phi) = (c_0, \phi_0)$ with $\phi_0 \in C^1((-\infty, 0])$ monotone if and only if **(J1)** holds. Moreover, when **(J1)** holds, there exists a unique solution pair, and $c_0 > 0$, $\phi_0'(x) < 0$.

(c) Sharper estimates of the spreading rate[4]

We will use the notation

$$\eta(t) \sim \xi(t) \;\text{ if and only if }\; c_1\xi(t) \le \eta(t) \le c_2\xi(t)$$

for some positive constants $c_1 \le c_2$.

Theorem 6 (Sharper estimates): Suppose **(J)** is satisfied, and spreading happens to the unique solution (u, g, h) of (4). If additionally

$$J(x) \sim |x|^{-\alpha} \;\text{ for } |x| \gg 1, \quad \left(\text{and so} \begin{cases} \textbf{(J)} \iff \{\alpha > 1\} \\ \textbf{(J1)} \iff \{\alpha > 2\} \end{cases} \right)$$

then

$$c_0t + g(t),\; c_0t - h(t) \sim 1 \qquad \text{if } \alpha > 3,$$

$$c_0t + g(t),\; c_0t - h(t) \sim \begin{cases} \ln t & \text{if } \alpha = 3, \\ t^{3-\alpha} & \text{if } 3 > \alpha > 2, \end{cases} \qquad \text{(shifts)}$$

$$-g(t),\; h(t) \sim \begin{cases} t\ln t & \text{if } \alpha = 2, \\ t^{\frac{1}{\alpha-1}} & \text{if } 2 > \alpha > 1. \end{cases} \qquad \text{(acceleration)}$$

[4] Y. Du, Wenjie Ni, Preprint 2020 (arXiv: 2010.01244).

Comparisons

- Under condition **(J)**,

$$\Big\{\text{accelerated spreading for (3)}\Big\} \Longleftrightarrow \Big\{J \text{ does not satisfy } (\mathbf{J}_{\rm thin})\Big\}.$$

$$\Big\{\text{accelerated spreading for (4)}\Big\} \Longleftrightarrow \Big\{J \text{ does not satisfy } \mathbf{(J1)}\Big\}.$$

For the corresponding local diffusion problems (1) and (2), accelerated spreading never happens.

- Under condition **(J)**,

$$(\mathbf{J}_{\rm thin}) \Longrightarrow \mathbf{(J1)} \left(\int_0^\infty J(x)e^{\lambda x}dx < \infty \Longrightarrow \int_0^\infty J(x)x dx < \infty\right)$$

$$\mathbf{(J1)} \not\Longrightarrow (\mathbf{J}_{\rm thin}).$$

Therefore, accelerated spreading happens less often in the nonlocal free boundary problem (4) than in the corresponding problem (3).

4. Extension to high dimensions with radial symmetry[5]

The radially symmetric version of (4) in $\mathbb{R}^N$ ($N \geq 2$) is

$$(6)\quad \begin{cases} u_t = d\int_{B_{h(t)}} J(|x-y|)u(t,|y|)dy - du + f(u), & t>0,\ x\in B_{h(t)},\\ u=0, & t>0,\ x\in\partial B_{h(t)},\\ h'(t) = \dfrac{\mu}{|\partial B_{h(t)}|}\displaystyle\int_{B_{h(t)}}\int_{\mathbb{R}^N\setminus B_{h(t)}} J(|x-y|)u(t,|x|)dydx, & t>0,\\ h(0)=h_0,\ u(0,|x|)=u_0(|x|), & x\in \overline{B}_{h_0}, \end{cases}$$

where $B_{h(t)} = \{x\in\mathbb{R}^N : |x| < h(t)\}$, and $u = u(t,|x|)$ is radially symmetric. The initial function u_0 satisfies

$$\begin{cases} u_0 \text{ is radial and continuous in } \overline{B}_{h_0},\\ u_0 > 0 \text{ in } B_{h_0},\ u_0 = 0 \text{ on } \partial B_{h_0}. \end{cases}$$

As before, for simplicity

$$f(u) = u(1-u).$$

[5] Y. Du and Wenjie Ni, Preprint, 2021 (arXiv:2102.05286)

For (6), our basic assumptions on the kernel function $J(|x|)$ are

(J): $J \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $J \geq 0$, $J(0) > 0$, $\int_{\mathbb{R}^N} J(|x|)dx = 1$.

For $r := |x|$ with $x \in \mathbb{R}^N$ and $\rho > 0$, define

$$\tilde{J}(r,\rho) = \tilde{J}(|x|,\rho) := \int_{\partial B_\rho} J(|x-y|)dS_y.$$

Then (6) can be rewritten into the equivalent form

$$(7)\quad \begin{cases} u_t(t,r) = d\displaystyle\int_0^{h(t)} \tilde{J}(r,\rho)u(t,\rho)d\rho - du + f(u), & t>0, r\in[0,h(t)),\\ u(t,h(t)) = 0, & t>0,\\ h'(t) = \dfrac{\mu}{h^{N-1}(t)}\displaystyle\int_0^{h(t)}\int_{h(t)}^{+\infty} \tilde{J}(r,\rho)r^{N-1}u(t,r)d\rho dr, & t>0,\\ h(0) = h_0,\ u(0,r) = u_0(r), & r\in[0,h_0]. \end{cases}$$

(Here a universal constant is absorbed by μ.)

Theorem 7 (Existence and uniqueness): Suppose $(\mathbf{J})$ is satisfied. Then problem (6), or equivalently (7), admits a unique positive solution (u, h) defined for all $t > 0$.

Theorem 8 (Spreading-vanishing dichotomy): Suppose $(\mathbf{J})$ is satisfied. Let (u, h) be the solution of (6). Then one of the following alternatives must occur :

(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$ and

$$\lim_{t\to\infty} u(t, |x|) = 1 \text{ locally uniformly in } \mathbb{R}^N,$$

(ii) Vanishing: $\lim_{t\to\infty} h(t) = h_\infty < \infty$ and

$$\lim_{t\to\infty} u(t, |x|) = 0 \text{ uniformly for } x \in B_{h(t)}.$$

Theorem 9 (Spreading-vanishing criteria): In Theorem 8,

(1) if $d \le f'(0) = 1$, then spreading always happens,

(2) if $d > f'(0) = 1$ then there exists $L_* > 0$ such that

  (i) for $h_0 \ge L_*$, spreading always happens,

  (ii) for $0 < h_0 < L_*$, there is $\mu_* > 0$ such that spreading happens if and only if $\mu > \mu_*$.

Here L_* is independent of u_0, but μ_* depends on u_0.

Spreading speed of (6)

We need to introduce the following function, which will determine the spreading speed. For any $\xi \in \mathbb{R}$, define

$$J_*(\xi) := \int_{\mathbb{R}^{N-1}} J(|(\xi, x')|)dx', \tag{8}$$

where $x' = (x_2, ..., x_N) \in \mathbb{R}^{N-1}$.

Condition **(J)** implies

$$\begin{cases} J_* \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ is nonnegative, even, } J_*(0) > 0, \\ \int_{\mathbb{R}} J_*(\xi)d\xi = \int_{\mathbb{R}^N} J(|x|)dx = 1. \end{cases}$$

Moreover,

$$J_*(\xi) = \omega_{N-1} \int_{|\xi|}^{\infty} J(r)r(r^2 - \xi^2)^{(N-3)/2}dr,$$

$$\int_0^\infty J_*(\xi)\xi d\xi = \frac{\omega_{N-1}}{N-1} \int_0^\infty J(r)r^N dr.$$

where ω_k denotes the area of the unit sphere in $\mathbb{R}^k$.

Theorem 10 (Spreading speed): Assume the conditions in Theorem 8 are satisfied, and spreading happens to (6). Then

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_0 & \text{if } J_* \text{ satisfies } \textbf{(J1)}, \\ \infty & \text{if } J_* \text{ does not satisfy } \textbf{(J1)}, \end{cases}$$

where c_0 is given by Theorem 5 with J replaced by J_*.

Rate of spreading with representative kernel functions

(i) Compactly supported kernels J always satisfy **(J1)**.
In dimension 1, when J has compact support, by a result in [Y. Du, Wenjie Ni, Preprint 2020 (arXiv: 2010.01244)],

$$c_0 t - h(t) \sim 1 \text{ for } t \gg 1.$$

In high dimension we have
Theorem 11 (Logarithmic shift): Suppose the conditions in Theorem 8 hold, and moreover the kernel function J has compact support. If spreading happens, then

$$c_0 t - h(t) \sim \ln t \text{ for } t \gg 1.$$

(ii) When $J(r) \sim r^{-\beta}$ for $r \gg 1$, then **(J1)** does not hold exactly when $\beta \in (N, N+1]$.

When $N = 1$, we know

$$h(t) \sim \begin{cases} t^{\frac{1}{\beta-1}} & \text{if } \beta \in (1,2), \\ t \ln t & \text{if } \beta = 2. \end{cases}$$

When $N \geq 2$, we have

Theorem 12 (Rate of accelerated spreading) Suppose the conditions in Theorem 8 are satisfied, and there exists $\beta \in (N, N+1]$ such that $J(r) \sim r^{-\beta}$ for all large r. If spreading happens, then for all large t,

$$h(t) \sim \begin{cases} t^{\frac{1}{\beta-N}} & \text{if } \beta \in (N, N+1), \\ t \ln t & \text{if } \beta = N+1. \end{cases}$$

Remarks and open problems:

- If $J(r) \sim r^{-\beta}$ for $r \gg 1$ and some $\beta > N+1$, then **(J1)** holds and $\lim_{t\to\infty} \frac{h(t)}{t} = c_0$ by Theorem 10.

  Open problem:

  What is the behaviour of $c_0 t - h(t)$ as $t \to \infty$?

- Conjectures:

  (a) In Theorem 11, $\displaystyle\lim_{t\to\infty} \frac{c_0 t - h(t)}{\ln t}$ exists.

  (b) In Theorem 12,

$$\begin{cases} \displaystyle\lim_{t\to\infty} \frac{h(t)}{t^{1/(\beta-N)}} \text{ exists when } \beta \in (N, N+1), \\ \displaystyle\lim_{t\to\infty} \frac{h(t)}{t \ln t} \quad \text{ exists when } \beta = N+1. \end{cases}$$

- Future questions: Consider singular kernel functions covering the fractional Laplacian case, and high dimension case without radial symmetry

5. Other extensions: Some epidemic and Lotka-Volterra systems in dimension 1

1. Y. Du, Wan-Tong Li, Wenjie Ni and Meng Zhao, *Finite or infinite spreading speed of an epidemic model with free boundary and double nonlocal effects*, submitted, 2020.
2. Y. Du, Mingxin Wang and Meng Zhao, *Two species nonlocal diffusion systems with free boundaries*, submitted, 2019. (arXiv:1907.04542)
3. Meng Zhao, Yang Zhang, Wan-Tong Li and Y. Du, *The dynamics of a degenerate epidemic model with nonlocal diffusion and free boundaries,* **J. Diff. Eqns**, 269(2020), 3347-3386.
4. Meng Zhao, Wan-Tong Li and Y. Du, *The effect of nonlocal reaction in an epidemic model with nonlocal diffusion and free boundaries*, **Comm. Pure Appl. Anal.**, 19(2020), 4599-4620.
5. Y. Du and Wenjie Ni, *Analysis of a West Nile virus model with nonlocal diffusion and free boundaries*, **Nonlinearity**, 33(2020), 4407-4448.

Thank You!