Blow-up of solutions of critical elliptic equations in three dimensions

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Asia–Pacific Analysis and PDE seminar, May 10, 2021

Joint work with T. König (Paris) and H. Kovarik (Brescia)

The problem of interest

Given $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ open and bounded, we are interested in

the behavior of solutions of

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \text{perturbation} & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

and the behavior of minimizers of

$$\inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2\, dx + \text{perturbation}}{\left(\int_\Omega |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}}}\,.$$

- Characteristic feature: scaling critical exponents $\frac{N+2}{N-2}$ / $\frac{2N}{N-2}$
- The embedding $H_0^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ is not compact.
- Problems with this feature appear in physics and geometry
- If perturbation=0, then no solution. Of interest to us are situations where the perturbation depends on a parameter $\epsilon \to 0$ and the solutions / minimizers $u = u_\epsilon$ converge weakly to zero in $H_0^1(\Omega)$ as $\epsilon \to 0$. Blow-up!

Some fundamental works from the eighties

Given $N \geq 3$, $\Omega \subset \mathbb{R}^N$ open and bounded and $V \in L^\infty(\Omega)$

Minimizers of
$$S_V = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega (|\nabla u|^2 + V u^2)\, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}}}.$$

are solutions of
$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} - Vu & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Are there minimizers for S_V, i.e., is the infimum attained?

Difficulty: Non-compactness of embedding $H^1_0(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$
Let $S =$ Sobolev constant (i.e. S_V with $V \equiv 0$ and $\Omega = \mathbb{R}^N$)

Theorem (Brézis–Nirenberg, Lieb (1983), Brézis (1985), Druet (2002))

If $N \geq 4$, the following are equivalent:

(i) S_V *is attained* (ii) $S_V < S$ (iii) $\inf_\Omega V < 0$.

If $N = 3$, the following are equivalent (with ϕ_V being defined soon):

(i) S_V *is attained* (ii) $S_V < S$ (iii) $\inf_\Omega \phi_V < 0$.

Some remarks

Theorem (Brézis–Nirenberg, Lieb (1983), Brézis (1985), Druet (2002))

If $N \geq 4$, the following are equivalent:

$$\text{(i) } S_V \text{ is attained} \qquad \text{(ii) } S_V < S \qquad \text{(iii) } \inf_\Omega V < 0\,.$$

If $N = 3$, the following are equivalent (with ϕ_V being defined soon):

$$\text{(i) } S_V \text{ is attained} \qquad \text{(ii) } S_V < S \qquad \text{(iii) } \inf_\Omega \phi_V < 0\,.$$

- There is a fundamental difference between dimensions $N \geq 4$ and $N = 3$
  $\inf_\Omega V < 0$ is a local condition, $\inf_\Omega \phi_V < 0$ is a nonlocal condition
- If $G_V(x, y) =$ Green's function for $-\Delta + V$ with Dirichlet boundary conditions, then

$$\phi_V(x) = -4\pi \lim_{y \to x} \left(G_V(x, y) - \frac{1}{4\pi} \, \frac{1}{|x - y|} \right)$$

- Case $N = 3$ is reminiscent of Schoen's work on the Yamabe problem
- This theorem has implications to the (non)existence of energy-minimizing solutions. By different means, sometimes one can show the (non)existence of other solutions.

Some more fundamental works from the eighties and nineties ($N \geq 4$)

Given $N \geq 4$ and $\Omega \subset \mathbb{R}^N$ open and bounded ($V \equiv -\epsilon$)

Problem 1. Consider solutions of

$$\begin{cases} -\Delta u_\epsilon = u_\epsilon^{\frac{N+2}{N-2}} + \epsilon u_\epsilon & \text{in } \Omega\,, \\ u_\epsilon > 0 & \text{in } \Omega\,, \\ u_\epsilon = 0 & \text{on } \partial\Omega\,. \end{cases}$$

with

$$\frac{\int_\Omega |\nabla u_\epsilon|^2\, dx}{\left(\int_\Omega u_\epsilon^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}}} \to S\,.$$

We know from BN that such solutions exist for all $\epsilon > 0$ and (easy) converge weakly to zero in $H^1_0(\Omega)$ and $u_\epsilon^{\frac{2N}{N-2}} \to S^{2/N}\delta_{x_0}$ in the sense of measures for some $x_0 \in \overline{\Omega}$.

Can one describe the behavior of the solutions u_ϵ in more detail?

Answer. Yes! Works by Budd, Atkinson–Peletier, Brézis–Peletier (rad. case, conjectures) and, finally, Han and Rey

Rather complete answer. We will not describe these results here in details.

More fundamental works from the eighties and nineties ($N \geq 4$), cont'd

Given $N \geq 4$ and $\Omega \subset \mathbb{R}^N$ open and bounded ($V \equiv -\epsilon$)

Problem 2. Consider minimizers u_ϵ of

$$\inf_u \frac{\int_\Omega (|\nabla u|^2 - \epsilon u^2)\, dx}{\left(\int_\Omega |u|^{\frac{2N}{N-2}}\, dx\right)^{\frac{N-2}{N}}}.$$

We know from BN that such minimizers exist for all $\epsilon > 0$ and (easy) converge weakly to zero in $H^1_0(\Omega)$ and $u_\epsilon^{\frac{2N}{N-2}} \to S^{2/N}\delta_{x_0}$ in the sense of measures for some $x_0 \in \overline{\Omega}$.

Can one describe the behavior of the minimizers u_ϵ in more detail?

Minimizers are solutions, so the previous analysis is applicable, but more precise questions

Answer. Yes! Works by Wei, Takahashi, see also F.–König–Kovarik (2020)

Rather complete answer. We will not describe these results here in details.

All this raises the question: What about $N = 3$?

Brézis–Peletier (1989) have a conjecture about this. This will be the main result today.

Critical potentials

According to Brezis–Nirenberg and Lieb in 3D we do not have minimizers for small V.

A function $a \in C(\overline{\Omega})$ is said to be critical (in the sense of Hebey–Vaugon (2001)) if $S_a = S$ and if for any continuous function $\tilde{a}$ on $\overline{\Omega}$ with $\tilde{a} \leq a$ and $\tilde{a} \not\equiv a$ one has $S_{\tilde{a}} < S_a$.

In the following, $N = 3$ and a critical. We consider either solutions $u = u_\epsilon$ of

$$\begin{cases} -\Delta u = 3u^5 - (a + \epsilon V)u & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad (\star)$$

satisfying

$$\frac{\int_\Omega |\nabla u_\epsilon|^2 \, dx}{\left(\int_\Omega u_\epsilon^6 \, dx\right)^{1/3}} \to S \qquad (\star\star)$$

or minimizers of

$$S_{a+\epsilon V} = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega (|\nabla u|^2 + (a + \epsilon V)u^2) \, dx}{\left(\int_\Omega u^6 \, dx\right)^{1/3}} \,.$$

Rough description of our main result

Let (u_ϵ) be a family of solutions to $(\star)$ satisfying $(\star\star)$.

- There is a concentration point $x_0 \in \Omega$
- Away from the concentration point, u_ϵ tends to zero and close to the concentration point, u_ϵ tends to infinity.
- More precisely, there is a localization length $\lambda_\epsilon^{-1} \approx \epsilon$ (typically)
- For $x \in \Omega \setminus \{x_0\}$ and with $G_a =$ the Green's function of $-\Delta + a$,

$$u_\epsilon(y) \approx \lambda_\epsilon^{-1/2}\, 4\pi\, G_a(y, x_0)\,.$$

- For $x \in \Omega$ with $|x - x_\epsilon| \lesssim \lambda_\epsilon^{-1}$ and $x_\epsilon \to x_0$,

$$u_\epsilon(y) \approx \left(\frac{\lambda_\epsilon}{1 + \lambda_\epsilon^2 |y - x_\epsilon|^2} \right)^{1/2}.$$

Difficulty: One would expect a localization length $\sim \epsilon^2 \ll \epsilon$ and maximum $\epsilon^{-1} \gg \epsilon^{-1/2}$. But there is a cancellation due to criticality and one needs to expand to higher precision and extract a subleading term close to x_ϵ.

Notation and assumptions

Let G_b be the Green's function of $-\Delta + b$ with Dirichlet boundary conditions and let

$$H_b(x,y) = -4\pi \left(G_b(x,y) - \frac{1}{4\pi} \frac{1}{|x-y|} \right)$$

Recall that $\phi_b(x) = \lim_{y\to x} H_b(x,x)$ and, since a is critical,

$$\inf_{\Omega} \phi_a = 0\,.$$

Assumptions.

(a) $\Omega \subset \mathbb{R}^3$ is a bounded, open set with C^2 boundary

(b) $a \in C^{0,1}(\overline{\Omega}) \cap C^{2,\sigma}_{\mathrm{loc}}(\Omega)$ for some $\sigma > 0$

(c) $V \in C^{0,1}(\overline{\Omega})$

(d) a is critical in Ω

(e) $a < 0$ in $\{\phi_a = 0\}$

(f) Any point in $\{\phi_a = 0\}$ is a nondegenerate critical point of ϕ_a, that is, for any x_0 with $\phi_a(x_0) = 0$, the Hessian $D^2\phi_a(x_0)$ does not have a zero eigenvalue

Comments. • One can show that $a \leq 0$ in $\{\phi_a = 0\}$, so (e) is not severe. In particular, it is satisfied if a is a constant.

• We believe that (f) is generically true. It is satisfied if $\Omega =$ ball and $a =$ const.

Notation

For $x \in \mathbb{R}^3$, $\lambda > 0$, let

$$U_{x,\lambda}(y) = \left(\frac{\lambda}{1 + \lambda^2 |y-x|^2} \right)^{1/2}$$

(so $-\Delta U_{x,\lambda} = 3 U_{x,\lambda}^5$ on $\mathbb{R}^3$) and let $PU_{x,\lambda}$ be its projection onto $H_0^1(\Omega)$, that is,

$$\Delta PU_{x,\lambda} = \Delta U_{x,\lambda} \quad \text{in } \Omega \,, \qquad PU_{x,\lambda} = 0 \quad \text{on } \partial\Omega \,.$$

Finally, let $\Pi_{x,\lambda}^\perp$ be the orthogonal (wrt $\int_\Omega \nabla u \cdot \nabla v \, dx$) projection onto the orthogonal complement of

$$\operatorname{span} \{ PU_{x,\lambda} \,,\ \partial_\lambda PU_{x,\lambda} \,,\ \partial_{x_1} PU_{x,\lambda} \,,\ \partial_{x_2} PU_{x,\lambda} \,,\ \partial_{x_3} PU_{x,\lambda} \} \,.$$

Asymptotic expansion of u_ε

Set $Q_V(x) = (4\pi)^2 \int_\Omega V(y) G_a(x,y)^2 \, dy$

Theorem (F.–König–Kovarik (2021))

Let (u_ϵ) be a family of solutions to $(\star)$ satisfying $(\star\star)$. Then there are sequences $(x_\epsilon) \subset \Omega$, $(\lambda_\epsilon) \subset (0,\infty)$, $(\alpha_\epsilon) \subset \mathbb{R}_+$ and $(r_\varepsilon) \subset T^\perp_{x_\varepsilon,\lambda_\varepsilon}$ such that

$$u_\epsilon = \alpha_\epsilon \left(PU_{x_\epsilon,\lambda_\epsilon} - \lambda_\epsilon^{-1/2} \Pi^\perp_{x_\epsilon,\lambda_\epsilon} (H_a(x_\epsilon,\cdot) - H_0(x_\epsilon,\cdot)) + r_\epsilon \right)$$

and a point $x_0 \in \Omega$ with $\phi_a(x_0) = 0$ and $Q_V(x_0) \le 0$ such that, along a subsequence,

$$\begin{aligned}
|x_\epsilon - x_0| &= o(\varepsilon^{1/2}) \,, \\
\phi_a(x_\epsilon) &= o(\epsilon) \,, \\
\lim_{\epsilon\to 0} \epsilon \, \lambda_\epsilon &= 4\pi^2 \, \frac{|a(x_0)|}{|Q_V(x_0)|} \,, \\
\alpha_\epsilon &= 1 + \frac{4}{3\pi^3} \, \frac{\phi_0(x_0) \, |Q_V(x_0)|}{|a(x_0)|} \, \epsilon + o(\varepsilon) \,, \\
\|\nabla r_\epsilon\|_2 &= \mathcal{O}(\epsilon^{3/2}) \,.
\end{aligned}$$

The Brézis–Peletier conjecture[1]

Corollary (F.–König–Kovarik (2021))

Let (u_ϵ) be a family of solutions to $(\star)$ satisfying $(\star\star)$. Then, with $(x_\epsilon) \subset \Omega$ and $(\lambda_\epsilon) \subset \mathbb{R}_+$ as in the previous theorem,

$$\lim_{\epsilon \to 0} \epsilon \, \|u_\varepsilon\|_\infty^2 = \lim_{\epsilon \to 0} \epsilon \, |u_\varepsilon(x_\varepsilon)|^2 = 4\pi^2 \, \frac{|a(x_0)|}{|Q_V(x_0)|}$$

and, uniformly for x in compacts of $\overline{\Omega} \setminus \{x_0\}$,

$$u_\varepsilon(x) = \lambda_\varepsilon^{-1/2} \, 4\pi \, G_a(x, x_0) + o(\lambda_\varepsilon^{-1/2}) \, .$$

- The theorem is proved (almost exclusively) using H^1 techniques. The corollary (which is an L^∞ assertion) is then derived using elliptic regularity (Moser iteration).
- Conversely, by Del Pino–Dolbeault–Musso (2004), for any x_0 as in the theorem and a constant, there is a solution of $(\star)$ blowing up at x_0 with some profile $U_{x_\epsilon, \lambda_\epsilon}$.
- It remains open whether these results hold without the nondegeneracy assumption.
- The case where $a(x) = 0$ for some $x \in \{\phi_a = 0\}$ remains open. Can one compute the asymptotics in this case? Or can one show that this case does not happen? We are grateful to H. Brézis for raising these questions.

[1] Strictly speaking, this is the translation of the third BP conjecture to our problem. The literal third BP conjecture (under a nondegeneracy condition on ϕ_V) is also proved in our paper.

Energy (quasi) minimizers

Theorem (F.–König–Kovarik (2021))

Assume that $\mathcal{N} := \{\phi_a = 0\} \cap \{Q_V < 0\} \neq \emptyset$. *Then* $S_{a+\epsilon V} < S$ *for all* $\epsilon > 0$ *and*

$$\lim_{\epsilon \to 0+} \frac{S_{a+\epsilon V} - S}{\epsilon^2} = -\left(\frac{3}{S}\right)^{\frac{1}{2}} \frac{1}{8\pi^2} \sup_{x \in \mathcal{N}} \frac{Q_V(x)^2}{|a(x)|} .$$

Moreover, let $(u_\epsilon) \subset H^1_0(\Omega)$ *be a family of nonnegative functions such that*

$$\lim_{\epsilon \to 0} \frac{\|u_\epsilon\|_6^{-2} \int_\Omega (|\nabla u_\epsilon|^2 + (a + \epsilon V) u_\epsilon^2)\, dx - S_{a+\epsilon V}}{S - S_{a+\epsilon V}} = 0 \qquad \text{and} \qquad \int_\Omega u_\epsilon^6\, dx = \left(\frac{S}{3}\right)^{\frac{3}{2}} .$$

Then one has the same decomposition of u_ϵ *as in the previous theorem*[a] *and, in addition,*

$$x_0 \in \mathcal{N} \text{ with } \frac{Q_V(x_0)^2}{|a(x_0)|} = \sup_{y \in \mathcal{N}} \frac{Q_V(y)^2}{|a(y)|} .$$

[a] except for the slightly weaker bound $\|\nabla r_\epsilon\| = o(\epsilon)$

- This theorem holds without the nondegeneracy assumption (f).
- It is interesting and potentially useful that the blow-up structure is valid for all 'quasi-minimizers' and independently of whether they satisfy an equation.
- The cancellation $S_{a+\epsilon V} = S + o(\epsilon)$ is Druet's theorem (a critical $\implies$ $\inf \phi_a = 0$).

Some general remarks

- There is a huge literature on blow-up analysis for elliptic equations with critical exponent. In some sense, our situation is the simplest blow-up situation, as it concerns single bubble blow-up of positive solutions in the interior. Much more refined blow-up scenarios have been studied, including, for instance, multi-bubbling, sign-changing solutions or concentration on the boundary under Neumann boundary conditions. Adimurthi, Bahri, Brendle, Brézis, Coron, del Pino, Dolbeault, Druet, Esposito, Grossi, Hebey, Han, Khuri, Li, Marques, Merle, Musso, Pacella, Peletier, Pistoia, Rey, Robert, Schoen, Struwe, Vaugon, Wei, Yadava and many, many more
- What makes this problem special is the extra cancellation (coming from $\phi_a(x_0) = 0$). Also the discussion of quasiminimizers seems to be nonstandard.
- The proofs of the solution and the minimizer theorems are based on an iterative improvement of the expansion. We need two iterations.
- The initial decomposition uses compactness and properties of the bubble.
- The first iteration is relatively standard (Rey, Exposito, ...): excluding boundary concentration and order sharp bound a-priori bound on

$$u_\epsilon - \alpha_\epsilon PU_{x_\epsilon,\lambda_\epsilon}\,.$$

- The second iteration is more subtle and problem-specific, having to deal with the cancellation: order sharp bound a-priori bound on

$$u_\epsilon - \alpha_\epsilon(PU_{x_\epsilon,\lambda_\epsilon} - \lambda_\epsilon^{-1/2}\Pi^\perp_{x_\epsilon,\lambda_\epsilon}(H_a(x,\cdot) - H_0(x,\cdot)))$$

  and, most importantly, finding the limiting behavior of λ_ϵ as ϵ.

Some general remarks, cont'd

- Difficulty in both cases: there are two scales, the global one (on which $u_\epsilon \to 0$) and the local one (on which $u_\epsilon \to \infty$)

$$u_\epsilon = \alpha_\epsilon \left(\underbrace{P}_{\text{global scale}} \underbrace{U_{x_\epsilon,\lambda_\epsilon}}_{\text{local scale}} - \lambda_\epsilon^{-1/2} \underbrace{\Pi^\perp_{x_\epsilon,\lambda_\epsilon}}_{\text{local scale}} \underbrace{(H_a(x,\cdot) - H_0(x,\cdot))}_{\text{global scale}} + \dots \right)$$

- In both cases, orthogonality conditions play an important role. Those can be maximally exploited in the H^1 setting.
- While the outcome of the iterations is the same, the methods are rather different in the solutions / quasiminimizer cases.
- In the solutions case, we use nine different Pohozaev identities.
  (Intuition: five unknown parameters $\lambda_\epsilon, \alpha_\epsilon, x_\epsilon$ each corresponds to one such identity, once in each iteration; minus one 'useless' identity due to cancellation)
  Which identity is useful depends on the available amount of a-priori information.
- In the quasiminimizer case, we use fundamentally minimality.
  Idea: Improvem't in energy $\implies$ improvem't in profile $\implies$ improvem't in energy
  Toy model:
  $$x^2 - 2ax = (x-a)^2 - a^2 \geq -a^2$$
  To be close to the minimum value $-a^2$, x needs to be close to a.
  Difficulty 1: The quadratic term is positive definite only under orthogonality cond's.
  Difficulty 2: Due to the different scales, it is not clear what is linear and quadratic

Summary

- We have discussed the resolution of the remaining Brézis–Peletier conjecture (1989) concerning the blow-up behavior of solutions to elliptic equations in 3D with critical exponent and critical lower order term.
- A characteristic cancellation in the problem requires a rather refined expansion.
- The cancellation, as well as the two-scale structure and the lack of coercivity, is most conveniently handled with orthogonality conditions and an H^1 analysis. Assertions in L^∞ are obtained only at the very end.

THANK YOU FOR YOUR ATTENTION!