Traveling wave solutions for the diffusive Lotka-Volterra system of 3 competing species

Chiun-Chuan Chen
National Taiwan University

Asia-Pacific Analysis and PDE Seminar
15 November, 2021

Outline

The results joint with or inspired by

M. Mimura, D. Ueyama, M. Tohma, L. Contento, T. Ogawa, LC Hung, CH Chang, ...

Part 1. Introduction

Important issue in ecology: biodiversity, coexistence of many species

Estimated numbers of current species on Earth in an AMS article by J. Malkevitch

- Animal species: 7.8 million of which 953,434 have been "described."
- Plant species: 300,000 with only 215,644 being "described."
- Fungi species: 610,000 of which 43,271 have been "described."

Biological interactions: competition, cooperation (mutualism), parasitism, commensalism, ...

We focus on competition.

Questions: relation between competition and coexistence

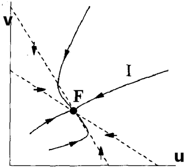
1. Does competition always reduce the chance of coexistence? Or sometimes it helps coexistence?
2. Can species coexist under strong competition?

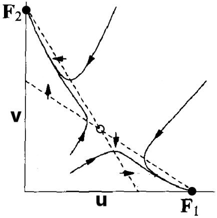
Lotka-Volterra competition system

ODE: $\begin{cases} u_t = u(r_1 - c_{11}u - c_{12}v) \\ v_t = v(r_2 - c_{21}u - c_{22}v) \end{cases}$

PDE: $\begin{cases} u_t = d_1 \Delta u + u(r_1 - c_{11}u - c_{12}v) \\ v_t = d_2 \Delta v + v(r_2 - c_{21}u - c_{22}v) \end{cases}$ on $\Omega \times (0, T)$

strong competition: only one species survives for almost all initial data
weak competition: 2 species coexist





Hofbauer and Sigmund

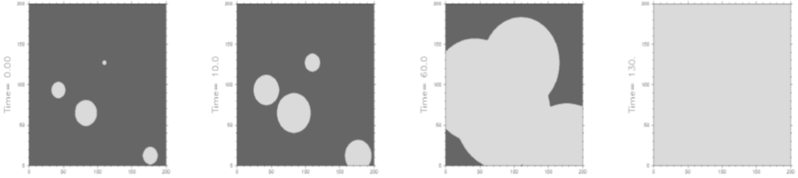
Strong competition in PDE case:

Theorem (K. Kishimoto, 1981) Consider the 2-species system with Neumann boundary condition on a convex domain. Then a stable steady state must be a constant.

Proof: 2-species system is monotone. Apply the comparison principle.

Theorem (H. Matano-M. Mimura, 1983) If the domain is far from being convex, then there exists a stable spatially-inhomogeneous equilibrium solution.

Mimura–Tohma's simulation



(a1) $t = 0$ (a2) $t = 3$ (a3) $t = 20$ (a4) $t = 50$

It is hard for 2 species to coexist in general.

3-species competition-diffusion system:

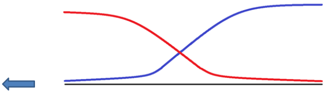
$$\begin{cases} u_t = d_1 \Delta u + u(r_1 - c_{11}u - c_{12}v - c_{13}w) \\ v_t = d_2 \Delta v + v(r_2 - c_{21}u - c_{22}v - c_{23}w) \\ w_t = d_3 \Delta w + w(r_3 - c_{31}u - c_{32}v - c_{33}w) \end{cases} \quad \text{on } \Omega \times (0,T)$$

- 2-species system is a monotone system. 3-species system is much more complicated: no comparison principle.

2-species traveling wave

1. Traveling waves are important in understanding the dynamical behavior.
2. The sign of the wave speed tells us which species is stronger in the PDE case. (Blue one is stronger)

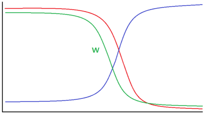
2-species traveling wave:



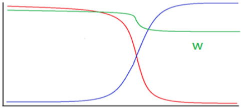
Theorem (Gardner, Conley-G, Kan-on, K-Fang 82, 84, 95, 96) Suppose that u and v are in strong competition. Then up to a translation, there exists a unique monotone traveling wave connecting $(0, \frac{r_2}{c_{22}})$ and $(\frac{r_1}{c_{11}}, 0)$. Moreover this wave is stable.

Question: What kind of waves can the 3-species system have?

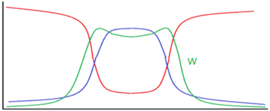
3-species traveling wave

- 

monotone wave

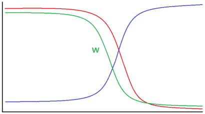


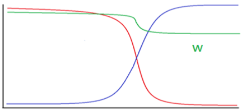
P. D. Miller, 1997

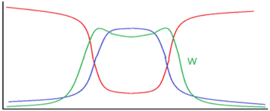


H. Ikeda, 2001

3-species traveling wave

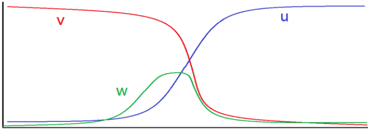
- 

monotone wave


P. D. Miller, 1997


H. Ikeda, 2001

- Question: Does there exist a type 4 wave?



Type 4

Consider the scenario:
Assume $v \succ w, w \succ u, u \succ v$. Initially v, w, u occupy the left, middle, and right. In the long run, can the 3 species develop into a wave with w in the middle?

Part 2. Existence of type 4 wave

I. Exact wave solutions for 2-species

- No good methods and techniques to tackle the problem. We turned to find exact solutions.

- M. Rodrigo and M. Mimura (2000, 2001): exact solutions of 2-species competition systems and other RD equations

$$\begin{cases} u_{xx} + su_x + u\,(1 - u - hv) = 0 \\ dv_{xx} + sv_x + rv(1 - ku - v) = 0 \quad \text{on } \mathbb{R} \end{cases}$$

$$\begin{cases} u(z) = \frac{1}{2}(1 + \tanh z) \\ v(z) = \frac{1}{4}(1 - \tanh z)^2, z = px - st, \end{cases}$$

where $d = \dfrac{r}{3h}, k = \dfrac{2-h}{r} + \frac{5}{3}, p = \dfrac{\sqrt{2h}}{4}, s = \dfrac{2-h}{4}.$

3-species competition-diffusion system:

$$\begin{cases} u_t = d_1 \, \Delta u + u(r_1 - c_{11}u - c_{12}v - c_{13}w) \\ v_t = \, d_2 \, \Delta v + v(r_2 - c_{21}u - c_{22}v - c_{23}w) \quad \text{on } \Omega \times (0,T) \\ w_t = d_3 \, \Delta w + w(r_3 - c_{31}u - c_{32}v - c_{33}w) \end{cases}$$

- 2-species: $v = G(u)$. 3-species: $v = G(u), w = H(u)$. (Unfortunately hard to apply to 3 species and get a solution.)
- 2-species system is a monotone system. 3-species system is much more complicated: no comparison principle.

II. Exact and numerical waves

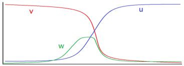
Chen, Hung, Mimura, and Ueyama 2012;
Chen, Hung, Mimura, Ueyama, and Tohma 2013;

Type 4 wave

- Assumed the solution = quadratic polynomials of $tanh$ and calculated by hand. The first example (similar to the below) was found:

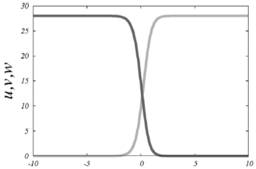
$$\begin{cases} u'' + su' + u(1 - u - \frac{8}{5}v - 2w) = 0 \\ v'' + sv' + v(1 - \frac{19}{25}u - v - \frac{1}{16}w) = 0 \quad \text{on } \mathbb{R} \\ w'' + sw' + w(1 - \frac{18}{25}u - \frac{8}{5}v - w) = 0 \end{cases}$$

has an exact TW solution

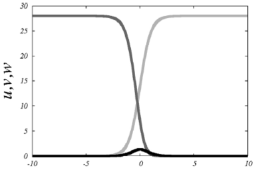


$$\begin{cases} u(z) = \frac{1}{2}(1 + \tanh(\frac{z}{5})) \\ v(z) = \frac{1}{4}(1 - \tanh(\frac{z}{5}))^2 \\ w(z) = \frac{4}{25}(1 - \tanh^2(\frac{z}{5})) \text{ with speed } s = \frac{11}{50} \end{cases}$$

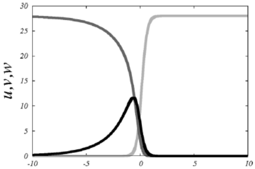
One can obtain more TW solutions if the coefficients satisfy suitable algebraic relations.



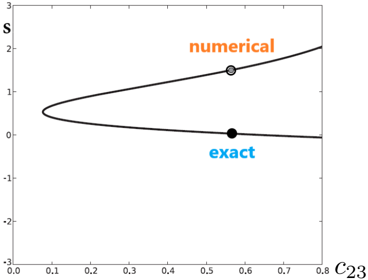
2-species wave, $s < 0$



exact wave, $s < 0$



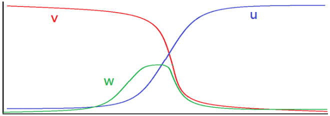
numerical wave, $s > 0$



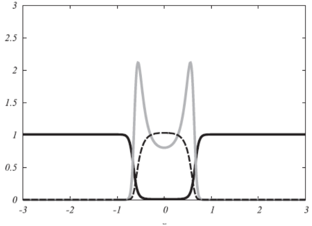
by AUTO numerical method

Two-peak wave

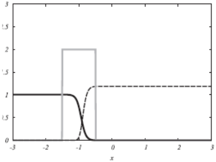
Type 4 wave:



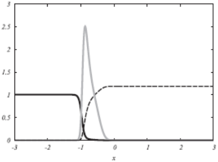
Chen-Hung-Mimura-Ueyama-Tohma 2013 (Ikeda wave with sharp peaks)



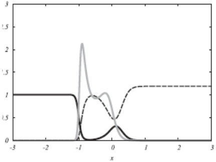
Appears in more complicated interactions:



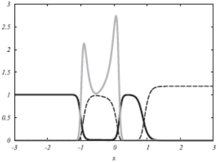
(a) $t = 0$



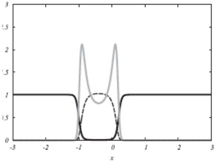
(b) $t = 0.01$



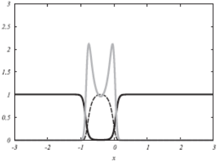
(c) $t = 0.05$



(d) $t = 0.07$



(e) $t = 0.20$



(f) $t = 100.$

Semi-exact solution:

Question: Is it possible to construct exact solutions for 2-peak waves?

- type 4 wave: (1) assume the wave is a simple polynomial of $\tanh$;
  (2) $\tanh' z =$ a polynomial of itself $= 1 - \tanh^2 z$.

Semi-exact solution:

Question: Is it possible to construct exact solutions for 2-peak waves?

- type 4 wave: (1) assume the wave is a simple polynomial of $\tanh$;
  (2) $\tanh' z =$ a polynomial of itself $= 1 - \tanh^2 z$.

- We failed to find a 2-peak exact solution through this ansatz.

Semi-exact solution:

Question: Is it possible to construct exact solutions for 2-peak waves?

- type 4 wave: (1) assume the wave is a simple polynomial of $\tanh$;
  (2) $\tanh' z =$ a polynomial of itself $= 1 - \tanh^2 z$.
- We failed to find a 2-peak exact solution through this ansatz.
- 2-peak wave: (1) the wave is a polynomial of some function $T(z)$;
  (2) $T'(z) =$ a simple polynomial of itself $= P(T(z))$;
  (3) Mathematica.

  Since $T(z)$ is defined implicitly by its ODE, such a wave is called a semi-exact solution.

$$\begin{cases} u_t = d_1 u_{xx} + (r_1 - a_1 u - b_{12} v - b_{13} w) u, \\ v_t = d_2 v_{xx} + (r_2 - b_{21} u - a_2 v - b_{23} w) v, & t > 0,\ x \in \mathbf{R}. \\ w_t = d_3 w_{xx} + (r_3 - b_{31} u - b_{32} v - a_3 w) w, \end{cases}$$

$$\begin{cases} d_1 = d_2 = d_3 = 1, \\ r_1 = 4(2+n)^2,\ r_2 = \frac{2(2+n)^2(12+7n)}{1+n},\ r_3 = \frac{2(2+n)^2(17+12n)}{1+n}, \\ a_1 = 4(1+n)(3+n),\ b_{12} = 4(-3+n)(1+n),\ b_{13} = 28(1+n), \\ b_{21} = 10(3+n)(4+3n),\ a_2 = 2(1+n)(12+7n),\ b_{23} = 28(1+n), \\ b_{31} = 10(3+n)(5+4n),\ b_{32} = 6(1+n)(5+4n),\ a_3 = 54(1+n). \end{cases}$$

The 3-species system has the solution

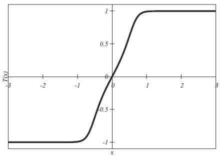
$$\begin{cases} u(x) = \dfrac{1}{1+n}[1 + (1+n)T^2(x)], \\ v(x) = [1 - T^2(x)]^2, \\ w(x) = \dfrac{1}{1+n}[1 + (1+n)T^2(x)][1 - T^2(x)]^2 \end{cases}$$

- $w = uv$, some hint?

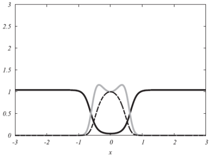
$$\begin{cases} \dfrac{d}{dx}T(x) = [1 - T^2(x)][1 + (1+n)T^2(x)], \\ T(0) = 0. \end{cases}$$

Profiles of T, u, v, w:

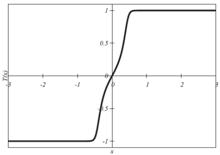
$$\frac{d}{dx}T = [1 - T^2][1 + (n+1)T^2].$$



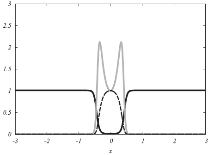
(a) $T(x)$ $(n = 3.01)$



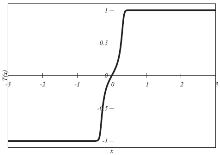
(b) $(u(x), v(x), w(x))$ $(n = 3.01)$



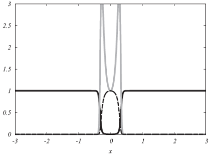
(c) $T(x)$ $(n = 10)$



(d) $(u(x), v(x), w(x))$ $(n = 10)$



(e) $T(x)$ $(n = 20)$



(f) $(u(x), v(x), w(x))$ $(n = 20)$

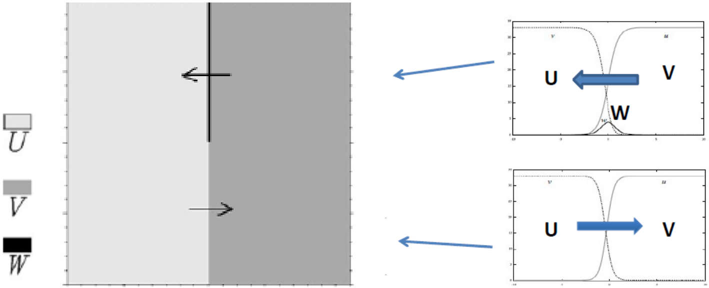
$$\begin{cases} u(x) = \dfrac{1}{1+n}[1 + (1+n)T^2(x)], \\ v(x) = [1 - T^2(x)]^2, \\ w(x) = \dfrac{1}{1+n}[1 + (1+n)T^2(x)][1 - T^2(x)]^2 \end{cases}$$

III. Spiral waves and dynamical patterns

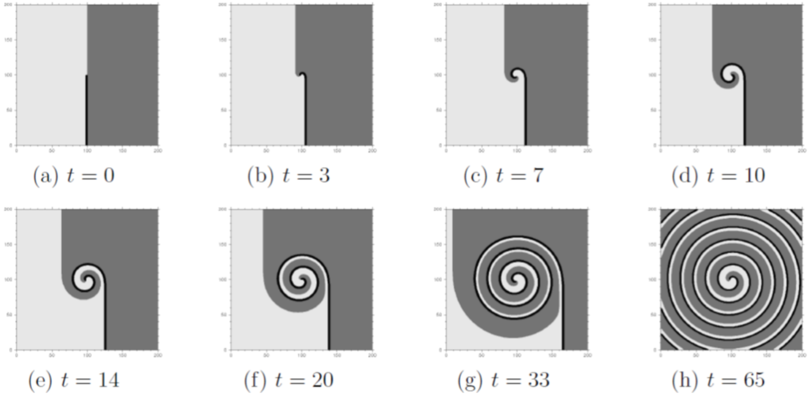
- applications of type 4 waves

Spiral wave

- u,v,w are symmetric (Ei–Ikota–Mimura, 1999): spiral wave
- u,v,w are asymmetric (Mimura–Tohma, 2015): use type 4 wave to construct spiral wave in more general situation



.....



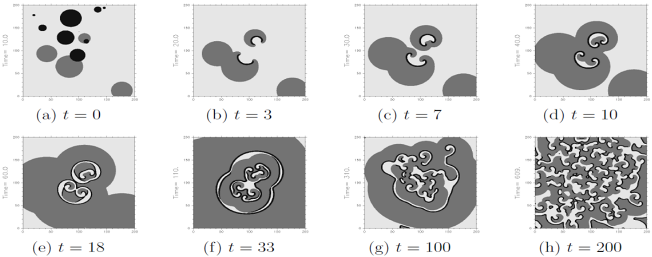
(a) $t = 0$ (b) $t = 3$ (c) $t = 7$ (d) $t = 10$

(e) $t = 14$ (f) $t = 20$ (g) $t = 33$ (h) $t = 65$

More dynamical patterns

- Mimura–Tohma (Ecol. Complexity, 2015), Contento–Mimura–Tohma (JJIAM, 2016): more waves, patterns and their interactions

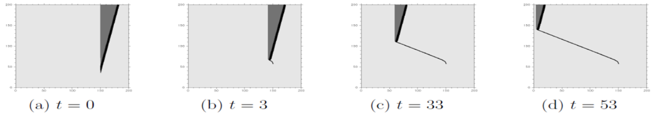
Dynamical coexistence pattern



(a) $t = 0$ (b) $t = 3$ (c) $t = 7$ (d) $t = 10$

(e) $t = 18$ (f) $t = 33$ (g) $t = 100$ (h) $t = 200$

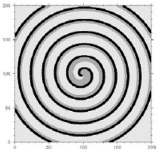
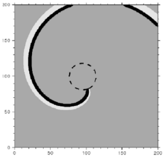
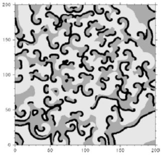
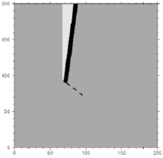
Wedge shape traveling wave



(a) $t = 0$ (b) $t = 3$ (c) $t = 33$ (d) $t = 53$

IV. Observation and conclusion

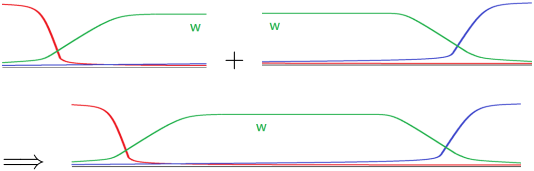
- 3 species have more chance than 2 species to coexist under strong competition.
  It seems coexistence is easier to happen as a dynamical pattern rather than a steady state.
- Under suitable conditions, even very strong competition alone can support coexistence.
- A 3 species system may produces "cyclic-like dominance" or other mechanisms to sustain the coexistence.
- Type 4 waves serve as a basic building block for complicated patterns.

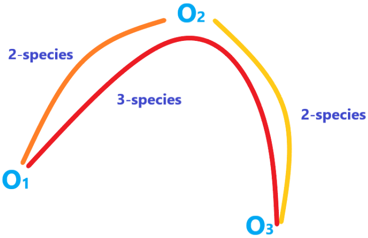
Question: How to construct more type 4 TW?

Part 3. Method of gluing bifurcation theory

Use the theory developed by Kokubu, Chow, Deng, Terman and Fiedler to construct a TW with long w in the middle.



$$u'' + su' + u(1 - u - c_{12}v - c_{13}w) = 0,$$
$$d_v v'' + sv' + v(r_2 - c_{21}u - v - c_{23}w) = 0,$$
$$d_w w'' + sw' + w(r_3 - c_{31}u - c_{32}v - w) = 0,$$



Let $\tau = (r_2, c_{21}, c_{12}, r_3, c_{31}, c_{32}, c_{23}, c_{13})$. For $\tau = \tau_o$, $\exists$ two 2-species waves with the same speed (τ_o, called the bifurcation point). Then perturb τ_o suitably to get a 3-species wave.

(H1) $h_i(z,\mu_0)$ are generic in the sense that, as $z \to -\infty$, $h_i(z,\mu_0)$ $(i=1,2)$ approaches $O_i(\mu_0)$ along the eigenspace associated with $\nu^i(\mu_0)$, and as $z \to +\infty$, it approaches $O_{i+1}(\mu_0)$ along the eigenspace associated with $-\rho^{i+1}(\mu_0)$.

(H2) The unstable manifold $W^u(O_i(\mu_0))$ $(i=1,2)$ and the stable manifold $W^s(O_{i+1}(\mu_0))$ have 1-dimensional intersection, i.e.,

$$\dim\{T_pW^u(O_i(\mu_0)) \cap T_pW^s(O_{i+1}(\mu_0))\} = 1$$

for all point $p \in h_i(z,\mu_0)$, where T_pW denotes the tangent space of the manifold W at p.

(H3) $W^u(O_i(\mu_0))$ $(i=1,2)$ is transversal to 4-dim ν-stable manifold $W^{\nu,s}(O_{i+1}(\mu_0))$ which is invariant and is tangent to the eigenspace corresponding to $\nu^{i+1}(\mu_0)$, $-\rho^i(\mu_0)$, $-\eta^i_j(\mu_0)$ $(j=1,2)$. Also, $W^s(O_{i+1}(\mu_0))$ is transversal to 4-dim $(-\rho)$-unstable manifold $W^{-\rho,u}(O_i(\mu_0))$ corresponding to $-\rho^i(\mu_0)$, $\nu^i(\mu_0)$, $\kappa^i_k(\mu_0)$ $(k=1,2)$.

(H4) The vectors given by the integrals

$$\mathbf{q}_1 := \int_{-\infty}^{\infty} \hat{q}^1(z)\partial_\mu F(h_1(z,\mu_0);\mu_0)dz, \quad \mathbf{q}_2 := \int_{-\infty}^{\infty} \hat{q}^2(z)\partial_\mu F(h_2(z,\mu_0);\mu_0)dz$$

are linearly independent, where up to a scalar multiple, $\hat{q}^i(z)$ is the unique non-trivial bounded solutions of $\hat{z}' = -\hat{z}\cdot\partial_Y F(h_i(z,\mu_0),\mu_0)$.

If $\nu^2(\mu_0) = \rho^2(\mu_0)$, then one additional hypothesis is needed:

(A1) The species are in strong competition.
(A2) The two species waves are stable in the 3-species system. (H. Ikeda)
(A3) The linearized behavior is dominated by the two species waves.
$O_1 : \lambda_u^{1,\pm}, \lambda_v^{1,\pm}, \lambda_w^{1,\pm};\ \ O_2 : \lambda_u^{2,\pm}, \lambda_v^{2,\pm}, \lambda_w^{2,\pm};\ \ O_2 : \ldots\ldots$

$$\left\{ \begin{array}{c} \lambda_u^{1,-} < \max\left\{\lambda_v^{1,-}, \lambda_w^{1,-}\right\} < 0 < \min\left\{\lambda_v^{1,+}, \lambda_w^{1,+}\right\} < \lambda_u^{1,+}, \\ \max\left\{\lambda_u^{2,-}, \lambda_v^{2,-}\right\} < \lambda_w^{2,-} < 0 < \lambda_w^{2,+} < \min\left\{\lambda_u^{2,+}, \lambda_v^{2,+}\right\}, \\ \lambda_v^{3,-} < \max\left\{\lambda_u^{3,-}, \lambda_w^{3,-}\right\} < 0 < \min\left\{\lambda_u^{3,+}, \lambda_w^{3,+}\right\} < \lambda_v^{3,+}, \\ \lambda_v^{1,-} \neq \lambda_w^{1,-},\ \lambda_v^{1,+} \neq \lambda_w^{1,+},\ \lambda_u^{3,-} \neq \lambda_w^{3,-},\ \lambda_u^{3,+} \neq \lambda_w^{3,+}. \end{array} \right\}$$

Theorem (CH Chang-C, JDDE 2021) Assume (A1)-(A3). Then $\exists$ Type 4 wave if the parameters are perturbed in suitable directions.

Example

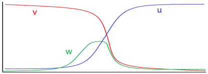
$d_v = d_w = 1, s = \frac{\alpha-1}{\sqrt{2(\alpha+1)}}, r_{2,0} = A(\alpha),\ \alpha > \frac{1}{3},\ A(\alpha) = \frac{\alpha(3\alpha-1)}{\alpha+1}$

$$\Phi_1(z) = \begin{pmatrix} 0 \\ v_0 \\ w_L \end{pmatrix}(z) = \begin{pmatrix} 0 \\ \frac{A(\alpha)}{4}\left(1-\tanh\sqrt{\frac{A(\alpha)+\alpha}{8}}z\right)^2 \\ \frac{\alpha}{4}\left(1+\tanh\sqrt{\frac{A(\alpha)+\alpha}{8}}z\right)^2 \end{pmatrix},$$

$$\Phi_2(z) = \begin{pmatrix} u_0 \\ 0 \\ w_R \end{pmatrix}(z) = \begin{pmatrix} \frac{1}{4}\left(1+\tanh\sqrt{\frac{1+\alpha}{8}}z\right)^2 \\ 0 \\ \frac{\alpha}{4}\left(1-\tanh\sqrt{\frac{1+\alpha}{8}}z\right)^2 \end{pmatrix}.$$

Part 4. Weak interaction

$$(\varepsilon\text{-P})\begin{cases} u_{zz}+su_z+u\,(1-u-c_{12}v-\varepsilon_1 w)=0,\\ d_2v_{zz}+sv_z+v\,(r_2-c_{21}u-v-\varepsilon_2 w)=0,\\ d_3w_{zz}+sw_z+w\,(r_3-c_{31}u-c_{32}v-c_{33}w)=0,\\ (u,v,w)(-\infty)=(0,r_2,0),\ (u,v,w)(\infty)=(1,0,0)\end{cases}$$



$$d_2=\frac{1}{3c_{12}},\quad c_{21}=\frac{6+5\,r_2-3\,c_{12}\,r_2}{3}$$

$$c_{31}=c_{32}r_2-\frac{r_2}{3d_2}+2,\ \ r_3=c_{32}r_2-\frac{(d_3+1)r_2}{6d_2}+1$$

$$r_2>3d_2,\ \ r_2d_3>|6d_2-r_2|,$$
$$c_{12}<\min\left\{\tfrac{2}{3}+\tfrac{2}{r_2},c_{32}+\tfrac{2}{r_2},\tfrac{2}{d_3+1}\left(c_{32}+\tfrac{2}{r_2}\right),\tfrac{c_{32}}{3d_3}\right\}.$$

Theorem (Chang-C-Hung-Mimura-Ogawa, Nonlinearity 2020)

If ε_1 and ε_2 are small, then $\exists$ a stable type 4 wave.

Proof: Prove the existence for the case $\varepsilon_1=\varepsilon_2=0$.

Then use perturbation argument.

Theorem 1.2. *Assume that* $c_{12}, d_2, r_2,$ *and* c_{21} *satisfy*

(a) $d_2 > 0$ *and* $c_{21} > r_2 > \dfrac{1}{c_{12}} > 0$; *and*

(b) *either* $c_{21} \geq b_0(r_2, d_2)$ *or* $c_{12} \geq c_0(r_2, d_2)$, *where*

$$b_0(r_2,d_2) = r_2 + d_2 + \begin{cases} 2r_2 d_2^2 \left(1 - \dfrac{1}{d_2}\right)\left(1 + \sqrt{1 + \dfrac{1}{r_2 d_2}}\right), & \text{if } d_2 \geq 1, \\ 2d_2 \left(\dfrac{1}{d_2} - 1\right)\left(\sqrt{2} - 1\right), & \text{if } 0 < d_2 < 1; \end{cases}$$

$$c_0(r_2,d_2) = \frac{1}{r_2} + \frac{1}{d_2} + \begin{cases} 2\left(1 - \dfrac{1}{d_2}\right)\left(\sqrt{2} - 1\right), & \text{if } d_2 \geq 1, \\ \dfrac{2}{r_2 d_2}\left(\dfrac{1}{d_2} - 1\right)\left(\sqrt{1 + r_2 d_2} + 1\right) & \text{if } 0 < d_2 < 1. \end{cases} \tag{1.12}$$

Then there exist positive d_3, r_3, c_{31}, c_{32}, c_{33} *and* δ *such that* (ε-*P*) *has a stable positive solution for* $0 \leq \varepsilon_1 < \delta$ *and* $0 \leq \varepsilon_2 < \delta$, *where* $\delta = \delta(c_{12}, d_2, r_2, c_{21}, d_3, r_3, c_{31}, c_{32}, c_{33})$. *Moreover from the construction of the solutions, one concludes that* dim $\Lambda = 11$.

$\Lambda = \{$ coefficients: which admit a type 4 TW $\}$

Question: What happens if the effect of w on u, v is not week?
($\varepsilon_1, \varepsilon_2$ are big)

Drift bifurcation + exact solution: Ei, Ikeda, Mimura and Ogawa.

- Find an exact 3-species solution for non-small $\varepsilon_1, \varepsilon_2$.
- Prove the first eigenfunction also has an exact form.
- Use bifurcation theory to construct solution theoretically and numerically.

3=2+1 reduction:

$$\begin{cases} u_t = d_1\, u_{xx} + u(r_1 - c_{11}u - c_{12}v - c_{13}w) \\ v_t = d_2\, v_{xx} + v(r_2 - c_{21}u - c_{22}v - c_{23}w) \quad \text{on } \mathbb{R} \\ w_t = d_3\, w_{xx} + w(r_3 - c_{31}u - c_{32}v - c_{33}w) \end{cases}$$

Important to estimate $p_{uv}(z) = c_{31}u(z) + c_{32}v(z)$

Question: Assume

$$\begin{cases} u_t = \Delta u + f(u,v) \\ v_t = d\,\Delta v + g(u,v) \quad \text{on } \Omega \\ \text{Neumann boundary condition.} \end{cases}$$

If $d \neq 1$, how to estimate $\alpha u + \beta v$?

For 1-dim traveling wave solution, such estimates can be obtained.

$$\begin{cases} u_{zz} + s\,u_z + u\,(1 - u - a_1\,v) = 0, \\ dv_{zz} + s\,v_z + \ v\,(1 - a_2\,u - v) = 0, \quad z \in \mathbb{R}. \\ (u,v)(-\infty) = (1,0), (u,v)(\infty) = (0,1). \end{cases}$$

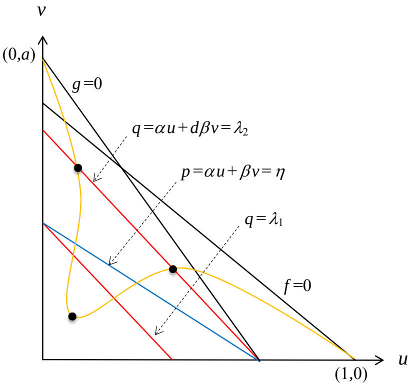
Theorem (LC Hung-C, JDE 2016) *Assume* $a_1, a_2 > 1$. *Then for* $\alpha > 0$ *and* $\beta > 0$,

$$\min\left[\frac{\alpha}{a_2}, \frac{\beta}{a_1}\right] \min[\frac{1}{d}, d] \leq \alpha u + \beta v \leq \max\left[\alpha, \beta\right] \max[\frac{1}{d}, d].$$

Proof: N-barrier method

$$q_{zz}(z) + s\,p_z(z) + \alpha\, f(u,v) + \beta\, g(u,v) = 0$$

$$q_z(z_2) - q_z(z) + s(p(z_2) - p(z)) + \int_z^{z_2} [\alpha\, f(u,v) + \beta\, g(u,v)] = 0$$



Application: Necessary condition for 3-secies wave

Theorem (Hung-C, JDE 2016) *Assume that* $d_1 \leq d_2$, u *and* v *competes strongly, and there exists a type 4 wave. Then*

$$r_3 > \min\left\{ \frac{\mathbf{c_{33}}}{\mathbf{c_{13}}} r_1,\ \frac{\mathbf{c_{33}}}{\mathbf{c_{23}}} r_2, \frac{d_1 \mathbf{c_{32}}}{d_2 \mathbf{c_{12}}} r_1, \frac{d_1 \mathbf{c_{31}}}{d_2 \mathbf{c_{21}}} r_2 \right\}$$

$$\begin{aligned} &u\left(r_1 - c_{11}u - \mathbf{c_{12}}v - \mathbf{c_{13}}w\right) \\ &v\left(r_2 - \mathbf{c_{21}}u - c_{22}v - \mathbf{c_{23}}w\right) \\ &w\left(r_3 - \mathbf{c_{31}}u - \mathbf{c_{32}}v - \mathbf{c_{33}}w\right) \end{aligned}$$

Part 5. Conclusion

- Type 4 waves serve as a basic building block for complicated patterns. One can use them to numerically construct spiral waves and new dynamic patterns
- 3 species have more chance than 2 species to coexist. It seems coexistence is easier to happen as a dynamical pattern rather than a steady state.
- Under suitable conditions, even very strong competition alone can support coexistence.
- Explanation for the existence/non-existence of exact solutions?
- How to prove the existence of spiral waves theoretically?

Thank you!