Current progress in higher-order curvature flow

Glen Wheeler

INSTITUTE FOR MATHEMATICS & ITS APPLICATIONS | UNIVERSITY OF WOLLONGONG AUSTRALIA

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The Plan

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  - Focus: Submanifolds without boundary, isotropic flows
  - General ideas: Existence, concentration-compactness, blowup, stability, convergence, **issues**

Three curvature flow

Surface diffusion flow. (Horizontal graphical) H^{-1}-gradient flow of area functional; Mullins '57 proposed:

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Chen's flow. Biharmonic heat flow for immersions; Bernard-W-Wheeler '19 proposed:

$$\partial_t f = -\Delta^2 f = -(\Delta H - H|A|^2)N \tag{3}$$

Issues and Challenges

Fun Fact

Higher-order PDE do not preserve positivity

Refs: Giga-Ito '98, Giga-Ito '99, Ito '99, Mayer-Simonett '00 and '03, Elliott-MaierPaape '01, Escher-Ito '05, Blatt '09, Blatt '10, W '13

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Glen will now draw a beautiful picture

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Higher-order PDE seem to behave, mostly, quite well

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- We need to understand stability in more ways

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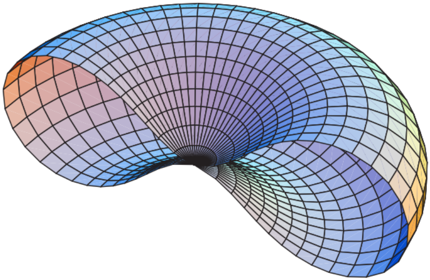
Glen will graffiti this



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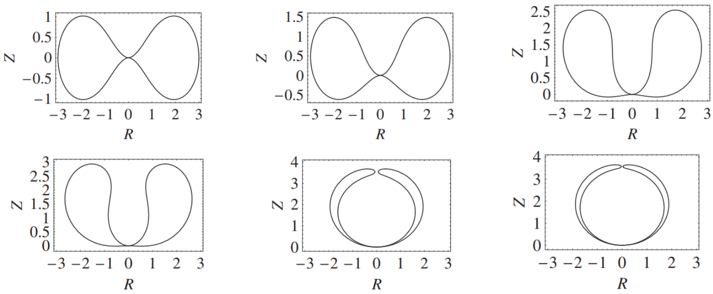

Refs: Gonzalez-Massari-Tamanini '83, Grüter '87, Morgan '00, Rosales '04, Castro-Guven '07,

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What we *can* do

Surface diffusion flow

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Concentration-compactness. $\exists \varepsilon_0, \delta_0 > 0$ s.t.

$$\sup_x \int_{f_0^{-1}(B_\rho(x))} |A|^2 \, d\mu < \varepsilon_0 \quad \Longrightarrow \quad T \geq \delta_0 \rho^4 \,,$$

with estimates.

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Giga's Question and **Chou's Conjecture** – more on these later.

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Most powerful results are 2D in one or two condimension.

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- **Convergence.** '$W^{2,2}$' nbhd of spheres in 2D Bernard-W-Wheeler '19 and 1D Cooper-W-Wheeler '19

Goals to keep in mind

Giga's Question (before '13)

Suppose $\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2$ is a curve diffusion flow with smooth initial data γ_0 that has the property:

$$\gamma(\cdot, t) \text{ is an embedding for each } t \in [0, T).$$

Must T then be ∞?

Chou's Conjecture '03

Suppose $\gamma : \mathbb{S} \times [0, T) \to \mathbb{R}^2$ is a curve diffusion flow with $T < \infty$ that satisfies the estimate

$$\|k\|_2^2(t) \le C(T-t)^{-1/4}\,, \tag{4}$$

for some $C \in \mathbb{R}$, and $t \in [0, T)$.

Then a parabolic rescaling (we assume the centre of mass of γ is the origin)

$$\eta(s,t) = (T-t)^{-\frac{1}{4}}\gamma(s,t)$$

about final time yields a self similar solution η to the curve diffusion flow, that is, η solves

$$\langle \eta, \nu^\eta \rangle = 4k_{ss}^\eta\,. \tag{Type I}$$

Thank you for your attention!