The Bernstein problem for parametric elliptic functionals

Connor Mooney

UC Irvine

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The Bernstein Problem

Theorem (Bernstein, 1915-17)

Assume $u \in C^2(\mathbb{R}^2)$ *solves the minimal surface equation*

$$div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Then u *is linear.*

- Different from linear case (many entire harmonic functions)

Bernstein Problem:

Prove the same result in higher dimensions, or construct a counterexample.

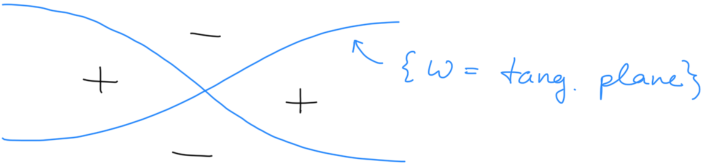
The Bernstein Problem

Solution to the Bernstein problem:

- $n = 2$ (Bernstein, 1915-17): Topological argument
- New proof (Fleming, 1962): Monotonicity formula, nontrivial solution in $\mathbb{R}^n \Rightarrow$ non-flat area-minimizing hypercone $K \subset \mathbb{R}^{n+1}$
- $n = 3$ (De Giorgi, 1965): $K = C \times \mathbb{R}$
- $n = 4$ (Almgren, 1966), $n \leq 7$ (Simons, 1968): Stable minimal cones are flat in low dimensions
- $n \geq 8$ (Bombieri-De Giorgi-Giusti, 1969): Counterexample!

The Bernstein Problem

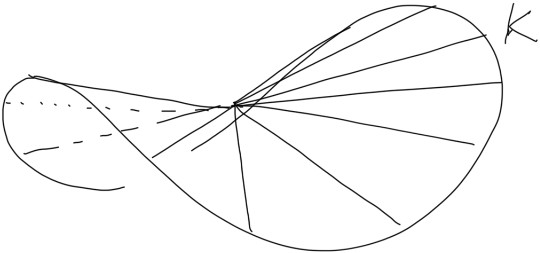
- $\det D^2 w < 0$ in $\mathbb{R}^2 \Rightarrow$ tang. planes to graph (w) disconn. it into ≥ 4 unbnd'd pieces



- "Cor:" $\left.\begin{array}{l} \underbrace{a_{ij}}_{\text{pos}}(x)\, w_{ij} = 0 \text{ in } \mathbb{R}^2 \\ w = o(|x|) \end{array}\right\} \Rightarrow w = \text{const.}$

- Apply to $w := \underbrace{\tan^{-1}(u_e)}_{\text{harmonic on graph}(u)}$

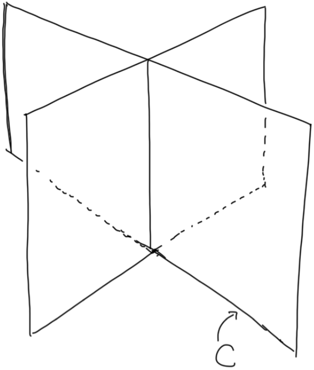
The Bernstein Problem



$K \subset \mathbb{R}^3$ cone + minimal $\Rightarrow$ flat

(only 1 nonzero curvature)

The Bernstein Problem



$K = \underbrace{C}_{} \times \mathbb{R}$

non-flat area-min. cone in $\mathbb{R}^n$

The Bernstein Problem

Bernstein's theorem generalizes to all dimensions with growth hypotheses:

- $|\nabla u| < C$ (De Giorgi-Nash, 1958)
- $u(x) < C(1 + |x|)$ (Bombieri-De Giorgi-Miranda, 1969)
- $|\nabla u(x)| = o(|x|)$ (Ecker-Huisken, 1990)

Some beautiful open problems:

- Do all entire solutions of the MSE have polynomial growth?
- Does there exist a nonlinear polynomial that solves the MSE?

Parametric Elliptic Functionals

Object of interest: $\Sigma \subset \mathbb{R}^{n+1}$ oriented hypersurface, minimizes

$$A_\Phi(\Sigma) := \int_\Sigma \Phi(\nu)\, dA.$$

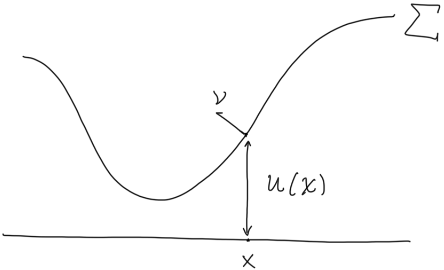
Here $\nu =$ unit normal, and Φ is 1-homogeneous, positive and $C^{2,\alpha}$ on $\mathbb{S}^n$, and $\{\Phi < 1\}$ uniformly convex ("uniform ellipticity")

E-L Equation: $\Phi_{ij}(\nu)II_{ij} = 0$ ("balancing of principal curvatures")

Φ-Bernstein Problem:

If Σ is the graph of a function $u : \mathbb{R}^n \to \mathbb{R}$, is it necessarily a hyperplane?

Φ-Bernstein Problem



$$\underbrace{\Phi_{ij}(\nu)}_{\text{Eigvals} \in [\lambda, \lambda^{-1}]} II_{ij} = 0$$

$$\Updownarrow$$

$$\varphi_{ij}(\nabla u)\, u_{ij} = 0,$$

$$\varphi(\cdot) = \Phi(-\cdot, 1)$$

(Ellipticity degen. as $|\nabla u| \to \infty$)

$$\int \Phi(\nu)\, dA = \int \Phi\left(\frac{-\nabla u, 1}{\sqrt{\cdot}}\right)\sqrt{\cdot}\, dx = \int \varphi(\nabla u)\, dx$$

Φ-Bernstein Problem

Positive results:

- $n = 2$ (Jenkins, 1961): ν is quasiconformal
- $n = 3$ (Simon, 1977): Regularity theorem of Almgren-Schoen-Simon (1977) for parametric problem
- $n \leq 7$ if $\|\Phi - 1\|_{C^{2,1}(\mathbb{S}^n)}$ small (Simon, 1977)
- $|\nabla u| < C$ (De Giorgi-Nash) or $|u(x)| < C(1 + |x|)$ (Simon, 1971)

Question: $4 \leq n \leq 7$???

Φ-Bernstein Problem

Theorem (M., '19)

There exists a quadratic polynomial on $\mathbb{R}^6$ whose graph minimizes A_Φ for a uniformly elliptic integrand Φ.

- Φ necessarily far from 1 on $\mathbb{S}^6$ (level sets "box-shaped")
- The analogous quadratic polynomial does not work in $\mathbb{R}^4$
- Open: $n = 4, 5$

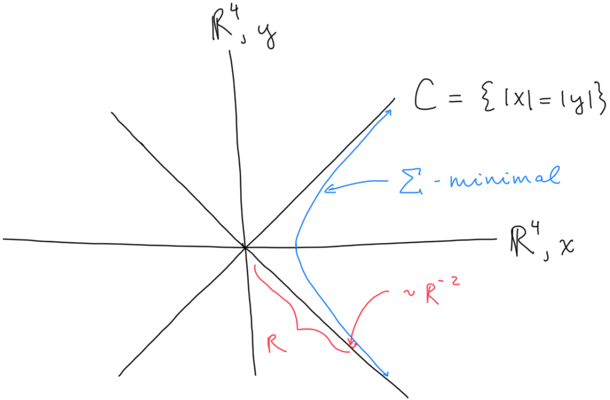
Φ-Bernstein Problem

Approach of Bombieri-De Giorgi-Giusti ($\Phi(x) = |x|$):

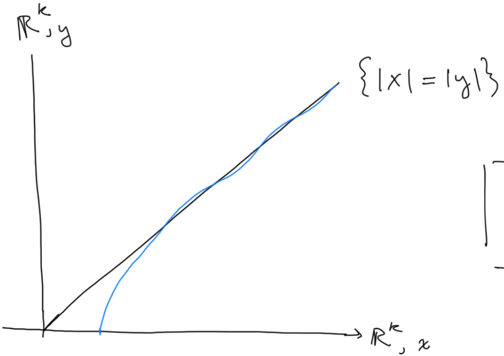
Let $(x, y) \in \mathbb{R}^8$ with $x, y \in \mathbb{R}^4$, and let $C := \{|x| = |y|\}$

- Find a smooth perturbation Σ of the Simons cone C, whose dilations foliate one side (ODE analysis)
- Notice that $\Sigma \sim \{r^3 \cos(2\theta) = 1\}$ far from the origin (here $r^2 = |x|^2 + |y|^2$, $\tan\theta = |y|/|x|$)
- Build global super/sub-solutions $\sim r^3 \cos(2\theta)$ in $\{|x| > |y|\}$ (**hard**), solve Dirichlet problem in larger and larger balls

Φ-Bernstein Problem



Φ-Bernstein Problem



The case $k \leq 3$

Φ-Bernstein Problem

Our approach: Fix u, build Φ

- Equation is $\varphi_{ij}(\nabla u)u_{ij} = 0$ (here $\varphi(p) := \Phi(-p, 1)$), rewrite in terms of Legendre transform u^* of u as

$$(u^*)^{ij}\varphi_{ij} = 0$$

  (a **linear hyperbolic eqn** for Φ)

- Let $(x, y) \in \mathbb{R}^{2k}$, $x, y \in \mathbb{R}^k$, $u = \frac{1}{2}(|x|^2 - |y|^2)$, $\varphi = \psi(|x|, |y|)$

  Equation becomes

$$\Box\psi + (k-1)\nabla\psi \cdot \left(\frac{1}{s}, -\frac{1}{t}\right) = 0$$

  in positive quadrant (here $|x| = s$, $|y| = t$, $\Box = \partial_s^2 - \partial_t^2$)

Φ-Bernstein Problem

The case $k = 3$ is special:

- Equation reduces to $\Box(st\,\psi) = 0$, so

$$\psi(s,\, t) = \frac{f(s+t) + g(s-t)}{st}$$

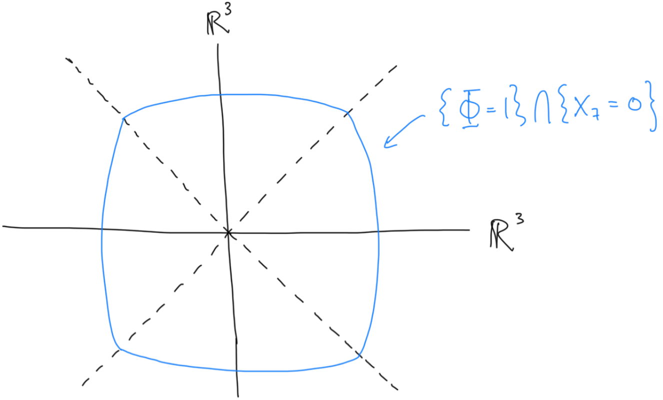
- Choose $f,\, g$ carefully s.t. Φ is uniformly elliptic (**tricky part**)

One choice of Φ is

$$\Phi(p,\, q,\, z) = \frac{\left((|p| + |q|)^2 + 2z^2\right)^{3/2} - \left((|p| - |q|)^2 + 2z^2\right)^{3/2}}{2^{5/2}|p||q|},$$

with $p,\, q \in \mathbb{R}^3$ and $z \in \mathbb{R}$.

Φ-Bernstein Problem



Remarks

Some remarks:

- There are many possible choices of Φ (perturb f, g)
- $\{u = const.\}$ minimize A_{Φ_0}, $\Phi_0 = \Phi|_{\{x_7=0\}}$ (homogeneity of u)
- The case $u = \frac{1}{2}(|x|^2 - |y|^2)$, $k = 2$: By above remark, $\{u = 1\}$ must minimize a uniformly elliptic functional. This is **false** when $k = 2$ (symmetries of u + ODE analysis)

  However, the cone $C := \{u = 0\} \subset \mathbb{R}^4$ does minimize a uniformly elliptic functional (Morgan, 1990)...

Current Work

(Joint with Y. Yang)

An approach in the case $n = 4$: combine the previous ones

1. Proof by "foliation" of Morgan's result:

   Calculations indicate can foliate sides of $C \subset \mathbb{R}^4$ by hypersurfaces that minimize uniformly elliptic functionals, look like level sets of γ-homogeneous functions with $\gamma \in (1, 2)$

2. Fix entire functions u on $\mathbb{R}^4$ that are asymptotically γ-homogeneous with $\gamma \in (1, 2)$, prove graphs minimize uniformly elliptic functionals

   (In dimension $n \geq 4$: same with $\gamma \in (1, n-2)$)

Current Work

(Joint with Y. Yang):

Controlled growth question:

- Positive result if $|\nabla u|$ grows slowly enough (e.g. $|\nabla u| = O(|x|^{\epsilon})$)?

Regularity of Φ:

- In above constructions, $\Phi \in C^{2,1}(\mathbb{S}^n)$. Can we make $\Phi \in C^{\infty}(\mathbb{S}^n)$? Analytic on $\mathbb{S}^n$?

Thank you!